AN ANALYTICAL METHOD TO DETERMINE THE MECHANICAL PROPERTIES
OF LINEAR VISCOELASTIC SOLIDS

By

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AN ANALYTICAL METHOD TO DETERMINE THE MECHANICAL PROPERTIES
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A new methodology has been developed to model the viscoelastic behavior of solids using a general spectrum function. Not all materials can be modeled using simple Kelvin-Voigt (K-V) or Maxwell elements where the viscoelastic parameters are constants. There is a need for a general spectrum function that can be used to model the Lamé’ functions which constitute all properties of interest. Thus far, there is no method like the one presented in this study that can determine the moduli of viscoelastic materials. This study develops a methodology by which the time dependent properties of homogeneous and non-homogeneous materials may be modeled.

Once the Lamé’ functions are determined, the Principle of Correspondence is applied to the elastic equations to determine the necessary properties. In uniaxial tension the time dependent strain, modulus, Poisson’s ratio, and compliance are determined. The time dependent deflection is determined for beams in flexure. Where applicable, parameters determined from the analytical model are compared to the available
experimental data. Good agreements are found between the analytical and experimental data sets.
DEDICATION

To: My Father
Acknowledgments

I would like to express my sincere gratitude to my major professor, Dr. James C. Newman, Jr. and to the members of my dissertation committee, Dr. Masoud Rais-Rohani, Dr. Richard Patton, Dr. Judith Schneider, and Dr. Mohsen Razzaghi. Their support, help, and insights are greatly appreciated.

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| λ<sub>e</sub>, µ<sub>e</sub> | Elastic Lame’ constants |
| λ(t), µ(t) | Memory functions or viscoelastic Lame’ functions |
| λ<sub>vc</sub>, µ<sub>vc</sub> | Kelvin-Voigt viscoelastic constants |
| φ(α) | Spectrum function; α = dummy variable |
| ρ | Density |
Symbol

\( \sigma \)  
Constant stress value

\( \nu_c \)  
Elastic Poisson’s ratio

\( \nu(t) \)  
Viscoelastic Poisson’s ratio

\( \zeta \)  
Laplace parameter

\( h \)  
Beam thickness

\( I \)  
Moment of inertia

\( L \)  
Beam length

\( m, n, n_o \)  
Viscoelastic constants

\( p \)  
Ply group

\( P \)  
Applied load

\( w \)  
Beam width

\( y_c \)  
Beam center deflection

\( e \)  
Strain tensor

\( f \)  
Body force vector

\( i_k \)  
Unit vector

\( I \)  
Identity tensor

\( T \)  
Stress tensor

\( u \)  
Displacement vector

\( x \)  
Cartesian coordinate vector
Symbol

\( \delta_{ij} \)  \hspace{1cm} \text{Kronecker symbol}

\( \delta(t) \)  \hspace{1cm} \text{Dirac Delta function}

\( \mathcal{L} \)  \hspace{1cm} \text{Laplace transform}

Tensor Operators:

\( \text{div} \)  \hspace{1cm} \text{Divergence}

\( \text{grad} \)  \hspace{1cm} \text{Gradient}
CHAPTER I
INTRODUCTION

This study deals with both homogeneous and non-homogeneous media in the field of linear viscoelasticity. This interest evolved due to a need to assess and predict the critical properties of composite materials under elevated temperatures where they behave in a viscoelastic manner. Environmental factors such as temperature play a key role in producing viscoelastic behavior in many materials. At elevated temperatures, both metals and polymer matrix composite materials depict varying degrees of viscoelastic behavior [1]. Although the field of linear viscoelasticity is well formulated, especially for homogeneous isotropic materials, a new methodology has been developed in this study from which vital material properties can be predicted.

Materials that exhibit characteristics of both elasticity and viscosity are called viscoelastic materials. One way to characterize these materials is to recognize the difference between elastic solids and viscous fluids in terms of storage and dissipation of mechanical energy. Elastic solids have the ability to store mechanical energy with no energy dissipation while viscous fluids are unable to store mechanical energy and they can provide continuous energy dissipation. Viscoelastic materials possess the capacity to store and dissipate energy at varying levels during the course of their loading histories. In elasticity, linear elastic materials obey Hooke’s Law in that the stress is proportional to
the strain and not the rate of strain, whereas in viscous fluids the opposite is true. Another characteristic of viscoelastic solids is that these materials have memories so that they “remember” past states of stress. For example, the deformation of a viscoelastic solid cannot be determined simply by its current state of stress; it is determined by its complete loading history [2,3]. A good overall discussion and introduction to viscoelasticity can be found in References [4,5,6].

Two of the early developers of the initial theories of viscoelasticity were Maxwell (1867) and Boltzmann (1874) and their theories were further enhanced by the likes of Kelvin (1875), Voigt (1889), and Volterra (1930), to name just a few. The constitutive equation of viscoelastic theory which relates stress and its time derivatives to strain and its time derivatives is expressed by the Boltzmann-Volterra Integral equation and simple models of viscoelasticity are represented by the Kelvin-Voigt and Maxwell models.

Experimentally, viscoelastic behavior is obtained through the use of either creep or stress relaxation tests. In creep, a constant stress is applied and the resulting displacement is measured, whereas in stress relaxation, a constant displacement is applied and the stress is monitored. A typical constant temperature creep curve is shown in 1.1. Similar curves are seen for metals, ceramics and polymers. The curve has three stages: primary, secondary, and tertiary. In the primary stage, the strain rate is high, but it is decreasing with time. During secondary creep, the strain rate is constant and in the tertiary phase it increases until failure occurs. Most creep design is based on the secondary stage and this phase is considered to be the most important due to this
component being experienced for the longest duration. This study also concentrates on
the secondary creep phase.

![Typical creep curve](image)

**Figure 1.1:** Typical creep curve.

To depict linear viscoelastic behavior, the most commonly used physical models
use springs which represent the Hookean elastic behavior and dashpots which model the
Newtonian viscous behavior. These elements are combined in series, resulting in a
Maxwell model or in parallel which results in the Kelvin-Voigt model. Shown in Figure
1.2 are the models with their corresponding creep compliance curves. Combination of
these elements results in various models that are used to describe viscoelastic behavior.
Detailed development of these simple models can be found in any book on linear
viscoelasticity, such as texts by Flugge [7], Christensen [2], Bland [8], and others.
Various methods to obtain the viscoelastic response have been reported in the literature. In experimental creep mechanics, different curve-fitting methods have been employed and from the fitted curves, viscoelastic constants have been extracted. Dutta and Hui [9] modified Findley’s power law equation for creep [10] to model the behavior of glass fiber reinforced polymer (GFRP) composites in tension and compression. Studies on the long term prediction of mechanical properties of polymer matrix composite materials have been performed by Tuttle and Brinson [11] using Shapery’s widely used non-linear viscoelastic model [12,13]. Since most of these solutions are expressed in series form as exponentials, a more comprehensive methodology covering both the theoretical and practical aspects must be developed.
Often the solutions to problems in viscoelasticity are hampered by mathematical complexities. A very powerful mathematical tool that is used in this study is the Principle of Correspondence, also known as the Correspondence Theorem or the Elastic-Viscoelastic Correspondence Principle. The premise of the principle is that assuming an elastic solution exists, its viscoelastic analog can be found by using the method of Laplace transform. This principle was first formulated by Alfrey [14] and was further used by Lee [15] for isotropic materials without temperature dependence and by Biot [16] for anisotropic materials. Using this principle, Schapery [17,18] correlated the viscoelastic properties of polymer matrix composite materials in terms of the constituent properties.

This study concentrates on linear viscoelastic behavior, which is characterized by the time dependent linear relationship between stress and strain. Linear theory is valid for small deformation analysis or when the strains and/or rates of strain are infinitesimal. In linear theory, both the properties of proportionality and superposition of responses are satisfied, Shapery [17]. For this research, in extending linear theory to layered media, the necessary properties are simply expressed by their effective counterparts. To establish the validity of using linear theory for anisotropic materials, Halpin and Pagano [19] used the Onsager reciprocity theorems and the Principle of Correspondence to demonstrate the symmetry in both the linear viscoelastic and the elastic cases.

Because the relevant analysis requires knowledge of the basic equations of elasticity and viscoelasticity, Chapter 2 is devoted to reviewing and stating the necessary equations using a continuum mechanics approach. Chapter 2 covers the basic
constitutive and equilibrium equations using the compact form of tensor notation. Also, the overall methodology of this study is mathematically developed in this chapter. The value of expressing material properties in their most basic form by using the notation of Lame’ constants is illustrated here.

The main effort and thrust of this research is contained in Chapter 3. Generally, viscoelastic behavior has been modeled using the elements of Kelvin-Voigt, Maxwell, or a combination of these models. Their use is limited in that these types of models are represented through the use of series and numerous terms are required to model the phenomenon appropriately. Eringen [20] suggests the use of memory functions by using a spectrum function to depict the behavior. However, Eringen does not suggest the manner in which this spectrum function should be chosen. In Chapter 3, a spectrum function that satisfies certain criteria is chosen to develop practically all the basic viscoelastic parameters as functions of time. This method allows for the mapping of a continuous distribution in a clean and smooth fashion. Once the spectrum function has been chosen properly, then the time dependent material properties of viscoelastic materials can be obtained. The development of the memory functions through the use of this spectrum function allows for a compact and complete representation of the pertinent viscoelastic functions.

The power of the Correspondence Principle is utilized in Chapter 4. This principle is applied to the constitutive equation for the uniaxial stress state to determine the strain as a function of time. It is also applied to the elastic deflection equation for a beam in flexure to obtain the time dependent deflection. Modulus, creep compliance, and
Poisson’s ratio are also obtained as functions of time. The flow chart in Figure B.1, Appendix B, shows the “road map” used in the general development of the methodology used in this study.

Finally, the analytical solutions are applied and the corresponding results are compared with experimental data. The test data was obtained from the University of Wyoming [21] and the Naval Surface Warfare Center (NSWC) - Carderock division [22]. Generally very good agreements are obtained between the analytical and experimental results. Throughout this study, focus is maintained on the applications of the results derived from the developed equations.
CHAPTER II

BASIC THEORETICAL FORMULATION

The goal of this chapter is to establish the basic equations of the mechanics of solids with elastic and memory effects. Eringen [20] and Timoshenko [23] give a detailed explanation of the formulation stated here. The pertinent equations are stated in vector-tensor forms because of their conciseness and applicability in general curvilinear coordinates. Also, the resulting terms from the following equations occur in subsequent discussions.

Beginning with the basic equation of dynamic equilibrium in continuum mechanics, the Navier equation, in tensor-vector form, is given as

\[ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho \mathbf{f} + \text{div} \mathbf{T} \]  

(2.1)

where \( \mathbf{u}(\mathbf{r}) \) is the displacement vector with \( \mathbf{r} = (x_i) \), \( \mathbf{T} \) is the stress tensor, \( \mathbf{f} \) is the body force vector representing force per unit mass, and \( \rho \) is the mass density. The formulation of \( \mathbf{T} \) comes from a constitutive equation that has been discussed in References [20] and [23]. For a linear elastic isotropic solid, the constitutive equation, also known as the generalized Hooke’s law, is

\[ \mathbf{T} = \lambda \text{div} \mathbf{u} \mathbf{I} + 2\mu \mathbf{e} \]  

(2.2)
where $\mathbf{I}$ is the unit tensor, $\mathbf{e}$ is the strain tensor, $\lambda_e$ and $\mu_e$ are the Lame` elastic constants.

Young’s modulus $E_e$, modulus of rigidity $G_e$, and Poisson’s ratio $\nu_e$, are related to Lame` constants as

$$
E_e = \frac{\mu_e (3\lambda_e + 2\mu_e)}{\lambda_e + \mu_e}
$$

$$
G_e = \mu_e
$$

$$
\nu_e = \frac{\lambda_e}{2(\lambda_e + \mu_e)}
$$

(2.3)

The infinitesimal strain tensor $\mathbf{e}$ in terms of the displacement vector $\mathbf{u(r)}$ is

$$
\mathbf{e} = \frac{1}{2} \left[ \text{grad} \mathbf{u} + (\text{grad} \mathbf{u})^T \right]
$$

(2.4)

The differential operator ‘grad’, (e.g., Warsi [24]), in terms of the Cartesian coordinates $x_i = (x_1, x_2, x_3)$ and the unit vectors $\mathbf{i}_k = (i_1, i_2, i_3)$ is

$$
\text{grad} \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_k} i_k
$$

$$
= \frac{\partial u_p}{\partial x_k} i_p i_k
$$

where repeated indices imply summation. Similarly, the transpose of $\text{grad} \mathbf{u}$ is

$$
(\text{grad} \mathbf{u})^T = i_k \frac{\partial \mathbf{u}}{\partial x_k}
$$

$$
= \frac{\partial u_p}{\partial x_k} i_k i_k = \frac{\partial u_k}{\partial x_p} i_p i_k
$$

Therefore,

$$
\mathbf{e} = e_{pk} i_p i_k
$$

$$
= \frac{1}{2} \left( \frac{\partial u_p}{\partial x_k} + \frac{\partial u_k}{\partial x_p} \right) i_p i_k
$$
where the Cartesian components of $e$ are

$$e_{pk} = \frac{1}{2}\left( \frac{\partial u_p}{\partial x_k} + \frac{\partial u_k}{\partial x_p} \right) \quad (2.5)$$

In the same manner, the unit tensor $I$ in Cartesian coordinates is

$$I = \delta_{mn} i_m i_n \quad (2.6)$$

Thus, in Cartesian coordinates, Eq. (2.2) is written as

$$T = \lambda_e \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu_e \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$= \lambda_e (e_{kk}) \delta_{ij} + 2\mu_e e_{ij} \quad (2.7)$$

It is easy to verify that from Eq. (2.2),

$$e = -\frac{\lambda_e}{2\mu_e} \text{div}(\text{div} u) I + \frac{1}{2\mu_e} T \quad (2.8)$$

Using Eq. (2.4) in Eq. (2.2) and then substituting the result in Eq. (2.1), the following is obtained.

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho f + \lambda_e \text{div}[(\text{div} u) I] + \mu_e \left[ \text{div} (\text{grad} u) + \text{div} (\text{grad} u)^T \right]$$

From Reference [24],

$$\text{div} (\text{grad} u)^T = \text{grad} (\text{div} u)$$

$$\text{div} [(\text{div} u) I] = \text{grad} (\text{div} u)$$

Using these definitions, the equation of dynamic equilibrium can also be expressed as

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho f + (\lambda_e + \mu_e) \text{grad} (\text{div} u) + \mu_e \left[ \text{div} (\text{grad} u) \right] \quad (2.9)$$
Equation (2.9) is the fundamental equation of linear elasticity. In Cartesian coordinates \((x_i)\), the equation is
\[
\rho \frac{\partial^2 u_i}{\partial t^2} = \rho f_i + (\lambda_c + \mu_c) \frac{\partial}{\partial x_i} \left( \frac{\partial u_p}{\partial x_p} \right) + \mu_c \nabla^2 u_i, \quad i = 1,2,3 \quad (2.10)
\]

For steady state, the time derivative term in Eq. (2.10) is zero. This equation has to be solved for \(u_i, i = 1,2,3\), under some prescribed boundary conditions. If \(S\) is the body surface, then [23]
\[
\mathbf{T} \cdot \mathbf{n} = \mathbf{s} \quad \text{or} \quad T_{ij} n_j = s_i \quad \text{on} \quad S_i
\]
\[
\mathbf{u} = \mathbf{w} \quad \text{or} \quad u_i = w_i \quad \text{on} \quad S_2
\]
where \(S = S_1 \cup S_2\), \(\mathbf{n}\) is the unit normal vector drawn outward on \(S\), and \(\mathbf{s}\) is the stress vector at the surface. Also, the solution needs the initial conditions,
\[
\left. u_i \right|_{t=0} = u_i^0(x) \quad \text{or} \quad \left. \frac{\partial u_i}{\partial t} \right|_{t=0} = U_i^0(x)
\]
\[
\left. \frac{\partial u_i}{\partial t} \right|_{t=0} = v_i^0(r) \quad \text{or} \quad \left. \frac{\partial u_i}{\partial t} \right|_{t=0} = v_i^0 (r)
\]
where
\[
\mathbf{r} = x_j \mathbf{i}_j
\]
For an isotropic elastic solid with memory, the constitutive equation, known as the Boltzmann-Volterra equation, e.g., [20] is
\[
\mathbf{T} = \lambda_c (\text{div} \mathbf{u}) \mathbf{I} + 2\mu_c \mathbf{e} + \int_{0}^{t} \left[ \lambda_v (t - \tau) \frac{\partial}{\partial \tau} (\text{div} \mathbf{u}) \mathbf{I} + 2\mu_v (t - \tau) \frac{\partial \mathbf{e}}{\partial \tau} \right] d\tau \quad (2.11)
\]
If the material is under stress before \(t = 0\), then the lower limit of integration should be \(-\infty\) instead of 0. In Eq. (2.11), \(\lambda_v(t)\) and \(\mu_v(t)\) are the memory functions. In this study, the
main goal is the determination of $\lambda_\nu(t)$ and $\mu_\nu(t)$ for both the isotropic and layered composite materials.

### 2.1 The Stress Tensor, $T$

Two important terms of physical importance can be extracted if the formula for $T$ as given in Eq. (2.11) is rewritten. First, the following functions must be introduced.

\[
K(t) = \lambda_c + \frac{2}{3} \mu_e + \lambda_\nu(t) + \frac{2}{3} \mu_\nu(t) \quad (2.12a)
\]

\[
G(t) = \mu_e + \mu_\nu(t) \quad (2.12b)
\]

where $K(t)$ is called the relaxation bulk modulus, and $G(t)$, the shear modulus. Now, Eq. (2.11) can be written as

\[
T = \mathbf{I}^i_0 K(t - \tau) \frac{\partial}{\partial \tau} (\text{div} \mathbf{u}) d\tau + \frac{1}{2} \mathbf{I} \frac{\partial \mathbf{e}}{\partial \tau} d\tau \quad (2.13)
\]

where

\[
\mathbf{e} = \mathbf{e} - \frac{1}{3} (\text{div} \mathbf{u}) \mathbf{I} \quad (2.14)
\]

In Cartesian component form, Eq. (2.13) and Eq. (2.14) become

\[
T_{ij} = \delta_{ij} \mathbf{I}^i_0 K(t - \tau) \frac{\partial e_{\nu}}{\partial \tau} d\tau + \frac{1}{2} \mathbf{I}^i_0 G(t - \tau) \frac{\partial \tilde{e}_{\nu}}{\partial \tau} d\tau \quad (2.15)
\]

where

\[
\tilde{e}_{ij} = e_{ij} - \frac{1}{3} e_{\nu} \delta_{ij} \quad (2.16)
\]

Along with Eq. (2.13), the expression for $\mathbf{e}$ is

\[
\mathbf{e} = \frac{1}{9} \mathbf{I}^i_0 B(t - \tau) \frac{\partial}{\partial \tau} (T : \mathbf{I}) d\tau + \frac{1}{2} \mathbf{I}^i_0 J(t - \tau) \frac{\partial \mathbf{T}}{\partial \tau} d\tau \quad (2.17)
\]
where $B(t)$ and $J(t)$ are called the creep bulk compliance and creep shear compliance, respectively. In Eq. (2.17),

$$\bar{T} = T - \frac{1}{3}(T : I)I$$

(2.18)

In Cartesian component form, using

$$T : I = T_{rr}$$

the strain can be expressed as

$$e_{ij} = \frac{1}{9} \delta_{ij} \int_0^t [B(t - \tau) \frac{\partial T_{rr}}{\partial \tau} d\tau + \frac{1}{2} J(t - \tau) \frac{\partial \bar{T}_{ij}}{\partial \tau} d\tau ] .$$

(2.19)

where

$$\bar{T}_{ij} = T_{ij} - \frac{1}{3} T_{nn} \delta_{ij}$$

(2.20)

Algebraic relations between $K$ and $B$, and $G$ and $J$, are given in terms of their Laplace transforms, cf. [20], [7], as

$$\bar{K}(\zeta) \bar{B}(\zeta) = \frac{1}{\zeta^2}$$

$$\bar{G}(\zeta) \bar{J}(\zeta) = \frac{1}{\zeta^2}$$

(2.21)

where $\zeta$ is the Laplace transform parameter, e.g.,

$$\bar{K}(\zeta) = \int_0^\infty e^{-\zeta t} K(t) dt$$

Taking the Laplace transforms of Eq. (2.13) and Eq. (2.17) while using the convolution theorem (Eq. (C.3), Appendix A) and Eq. (2.21), it can be shown that Eq. (2.13) and Eq. (2.17) are mutually consistent.
Two important functions of the Laplace transform parameter $\zeta$ which frequently appear in this study are

\begin{align}
\lambda(\zeta) &= \lambda_e + \zeta \lambda_v(\zeta) \tag{2.22a} \\
\mu(\zeta) &= \mu_e + \zeta \mu_v(\zeta) \tag{2.22b}
\end{align}

These functions appear naturally if the Laplace transform of the stress tensor, $T$ as given in Eq. (2.11), is taken. Thus,

\begin{align}
\overline{T} &= \lambda_e \left( \text{div } \overline{u} \right) I + 2\mu_e \overline{e} + \zeta \lambda_v \left( \text{div } u \right) I + 2\zeta \mu_v \overline{e} - g_i \\
&= \left( \lambda_e + \zeta \lambda_v \right) \left( \text{div } \overline{u} \right) I + 2(\mu_e + \zeta \mu_v) \overline{e} - g_i \tag{2.23a}
\end{align}

where

\[ g_i = \lambda_v(\zeta) \left( \text{div } u^o \right) I + 2\mu_v(\zeta) \overline{e}^o \]

and

\[ u^o = u_{|t=0} \quad e^o = e_{|t=0} \]

Equation (2.23a) has been obtained under the condition that for $t < 0$, $e = 0$. Introducing the notation of Eq. (2.22), the following is obtained.

\[ \overline{T} = \lambda(\zeta) \left( \text{div } \overline{u} \right) I + 2\mu(\zeta) \overline{e} - g_i \tag{2.23b} \]

It must be emphasized here that $\lambda(\zeta)$ and $\mu(\zeta)$ are not the Laplace transforms of functions but they are only functions of the transform parameter $\zeta$. The similarity of forms between Eq. (2.2) and Eq. (2.23b) in the first two terms is germane to the “Principle of Correspondence” to be discussed later.
2.2 Moduli and Compliance

The formulas for the Laplace transform of the relaxation Young’s modulus $E(t)$ and Poisson’s ratio $\nu(t)$ from Ref. [20] and [7] must be stated:

\[
\bar{E}(\zeta) = \frac{9G(\zeta)K(\zeta)}{3K(\zeta) + G(\zeta)} \quad (2.24)
\]

\[
\bar{\nu}(\zeta) = \frac{3K(\zeta) - 2G(\zeta)}{6K(\zeta) + 2G(\zeta)} \cdot \frac{1}{\zeta} \quad (2.25)
\]

Taking the Laplace transforms of Eq. (2.12a) and Eq. (2.12b) while using Eq. (2.22), and from Eq. (2.24) and Eq. (2.25), it can be shown that

\[
\zeta \bar{E}(\zeta) = \mu(\zeta) \left[ \frac{3\lambda(\zeta) + 2\mu(\zeta)}{\lambda(\zeta) + \mu(\zeta)} \right] \quad (2.26)
\]

\[
\zeta \bar{\nu}(\zeta) = \frac{\lambda(\zeta)}{2[\lambda(\zeta) + \mu(\zeta)]} \quad (2.27)
\]

Similarly, using Eq. (2.21), the following is obtained.

\[
\zeta \bar{B}(\zeta) = \frac{3[1 - 2\zeta \bar{\nu}(\zeta)]}{\zeta \bar{E}(\zeta)} \quad (2.28)
\]

\[
\zeta \bar{J}(\zeta) = \frac{2[1 + \zeta \bar{\nu}(\zeta)]}{\zeta \bar{E}(\zeta)} \quad (2.29)
\]

From Eq. (2.28) and Eq. (2.29) it is seen that both $B(t)$ and $J(t)$ cannot be expressed in simple forms as is the case with $K(t)$ and $G(t)$ and given in Eq. (2.12). Although the expressions for $\bar{E}(\zeta)$ and $\bar{\nu}(\zeta)$ from Eq. (2.26) and Eq. (2.27) may be substituted into Eq. (2.28) and Eq. (2.29) to express these quantities in terms of $\lambda(\zeta)$ and $\mu(\zeta)$, the forms are difficult to invert to obtain $B(t)$ and $J(t)$. Instead, these functions are expressed by the inversion of their transforms, Eq. (2.28) and Eq. (2.29).
2.3 A Special Form of the Constitutive Equation

Let $\lambda_v(t)$ and $\mu_v(t)$ in Eq. (2.11) be expressed in terms of the Dirac delta function (Eq. (A1), Appendix A) as

$$
\begin{align*}
\lambda_v(t) &= \lambda_{vc} \delta(t) \\
\mu_v(t) &= \mu_{vc} \delta(t)
\end{align*}
$$

(2.30)

where $\lambda_{vc}$ and $\mu_{vc}$ are constants. The constitutive Eq. (2.11) becomes

$$
T = \lambda_v (\text{div} \, u) \mathbf{I} + 2\mu_v \mathbf{e} + \int_0^t \left[ \lambda_{vc} \delta(t - \tau) \frac{\partial}{\partial \tau} (\text{div} \, u) \mathbf{I} + 2\mu_{vc} \delta(t - \tau) \frac{\partial \mathbf{e}}{\partial \tau} \right] d\tau
$$

Using (Eq. (A5), Appendix A), a special form of the constitutive equation is obtained as

$$
T = \left[ \lambda_e (\text{div} \, u) + \lambda_{vc} \frac{\partial}{\partial t} (\text{div} \, u) \right] \mathbf{I} + 2\left[ \mu_e \mathbf{e} + \mu_{vc} \frac{\partial \mathbf{e}}{\partial t} \right]
$$

(2.31)

Materials satisfying Eq. (2.31) are said to conform to the Kelvin-Voigt (K-V) model. In Cartesian component form Eq. (2.31) is

$$
T_{ij} = \left[ \lambda_e e_{ir} + \lambda_{vc} \frac{\partial e_{ir}}{\partial t} \right] \delta_{ij} + 2\left[ \mu_e e_{ij} + \mu_{vc} \frac{\partial e_{ij}}{\partial t} \right]
$$

(2.32)

The Laplace transform of Eq. (2.31) is similar to Eq. (2.23), but for a K-V model,

$$
\lambda(\zeta) = \lambda_e + \zeta \lambda_{vc}
$$

(2.33a)

$$
\mu(\zeta) = \mu_e + \zeta \mu_{vc}
$$

(2.33b)

Therefore, for a K-V model, $\lambda(\zeta)$ and $\mu(\zeta)$ are linear functions of $\zeta$, and $\lambda_{vc}$ and $\mu_{vc}$ are the material constants. It is useful to compare Eq. (2.22) and Eq. (2.33).
2.4 The Theorem of Correspondence

As has been stated in very simple terms by Eringen [20], for a linear viscoelastic problem with time independent boundaries, the solution can be obtained from the solution of the corresponding problem in linear elasticity. The meaning of this simple statement becomes clear if the Laplace transform of each term of the Navier equation Eq. (2.1) is taken, where for a function \( \lambda(r,t) \), the Laplace transform is

\[
\mathcal{L}(\lambda) = \bar{\lambda} = \int_0^\infty e^{-\zeta \psi} \lambda(r,t) \, dt
\]

Thus,

\[
\mathcal{L} \left( \rho \frac{\partial^2 u}{\partial t^2} \right) = \mathcal{L}(\rho f) + \mathcal{L} \left( \text{div} \, T \right)
= \mathcal{L}(\rho f) + \text{div} \left( \mathcal{L} \left( T \right) \right)
\tag{2.34}
\]

Now,

\[
\mathcal{L} \left( \rho \frac{\partial^2 u}{\partial t^2} \right) = - \frac{\partial u}{\partial t} \bigg|_{t=0} - \zeta u \bigg|_{t=0} + \zeta^2 \bar{u}
\tag{2.35a}
\]

Further, from Eq. (2.23) for a viscoelastic material,

\[
\text{div} \, \bar{T} = \lambda(\zeta) \text{div} \{(\text{div} \, \bar{u}) I\} + 2\mu(\zeta) \text{div} \, \bar{e} - \bar{g}_2
= \lambda(\zeta) \text{grad} \text{div} \, \bar{u} + 2\mu(\zeta) \text{div} \, \bar{e} - \bar{g}_2
\tag{2.35b}
\]

where

\[
\bar{g}_2 = \bar{\lambda}_v(\zeta) \text{grad} \text{div} \, \bar{u} + 2\bar{\mu}_v(\zeta) \text{div} \, \bar{e}^o
\]

Substituting Eq. (2.35) in Eq. (2.34), the following is obtained.

\[
\lambda(\zeta) \text{grad} \text{div} \, \bar{u} + 2\mu(\zeta) \text{div} \, \bar{e} + \rho \left\{ F + v^o + \zeta u^o - \zeta^2 \bar{u} \right\} = 0
\tag{2.36}
\]
where

\[ v^o = \left. \frac{\partial u}{\partial t} \right|_{t=0} \quad u^o = \left. u \right|_{t=0} \]

are the specified initial values. Further, \( F \) is defined by the equation

\[ \rho F = \rho \bar{f} - \mathbf{g}_2 \]

Substituting Eq. (2.2) into Eq. (2.1), the corresponding problem in linear elasticity is stated as

\[ \lambda_e \text{grad}(\text{div} \mathbf{u}) + 2\mu_e \text{div} \mathbf{e} + \rho \mathbf{f} - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0 \]  \hspace{1cm} (2.37)

Taking the Laplace transform of Eq. (2.37), the following is obtained.

\[ \lambda_e \text{grad}(\text{div} \mathbf{\bar{u}}) + 2\mu_e \text{div} \mathbf{\bar{e}} + \rho \mathbf{\bar{f}} + v^o + \zeta u^o - \zeta^2 \mathbf{\bar{u}} = 0 \]  \hspace{1cm} (2.38)

A comparison of Eq. (2.36) and Eq. (2.38) shows the similarity of forms; the difference being that for viscoelastic problem \( \lambda_e \) and \( \mu_e \) are replaced by \( \lambda(\zeta) \) and \( \mu(\zeta) \), respectively, and \( \bar{f} \) is replaced by \( F \). The same similarity appears in the conditions applied at the time independent boundaries. This is the essence of the Theorem of Correspondence. A formal statement of the theorem is as follows:

**Theorem of Correspondence:** If the boundary conditions are specified on time independent bounding surfaces, then the solution of the Laplace transform of linear isotropic viscoelasticity is identical to the solution of the problem of linear isotropic elasticity with \( \lambda_e \) replaced by \( \lambda(\zeta) \) and \( \mu_e \) replaced by \( \mu(\zeta) \), and the appropriate boundary conditions replaced by their transforms.
CHAPTER III
FORMULATION OF THE VISCOELASTIC PARAMETERS
USING A SPECTRUM FUNCTION

There are very few materials which follow the viscoelastic behavior as depicted by Kelvin-Voigt models (Section 2.3). Therefore, the main aim of this chapter is to formulate the functional forms of \( \lambda_v(t) \) and \( \mu_v(t) \) which cover a wide range of viscoelastic materials. Embedded in the proposed model are certain temperature dependent parameters, which when modeled closely, produce very favorable comparisons with the experimental data on strain and other response parameters.

3.1 Functional Formulation of the Viscoelastic Parameters

As stated in Ref. [20], one choice can be in the form of a Prony series, e.g.,

\[
\mu_v(t) = \sum_{n=0}^{\infty} \mu_n e^{-k_n t}, \quad k_n > 0
\]

where \( \mu_n \) and \( k_n \) are constants. The difficulty with such a representation is that the computational work increases tremendously as the number of terms increases. Instead of series representations for \( \lambda_v(t) \) and \( \mu_v(t) \), Eringen [20] suggests the use of a continuous spectrum function \( \varphi(\alpha) \) which appears in the formulation as

\[
\lambda_v(t) = -\lambda_v \int_0^{\infty} \varphi(\alpha) \left[ 1 - e^{-\alpha t} \right] d\alpha \quad (3.1a)
\]
\[ \mu_\alpha(t) = -\mu_\infty \int_0^\infty \phi(\alpha) \left[ 1 - e^{-\alpha t} \right] d\alpha \] \quad (3.1b)

where \( \phi(\alpha) \) is called the relaxation spectrum having the properties

\[ \phi(0) \geq 0, \quad \phi(\infty) \leq 1 \]

In this research Eq. (3.1) is taken for general viscoelastic materials without imposing any limiting requirements. On the other hand for a K-V model, taking a lead from Eq. (2.30),

\[ \lambda_\alpha(t) = -\left[ \lambda_\infty \theta \lim_{\alpha \to 0} \phi(\alpha) \left[ 1 - e^{-\alpha t} \right] d\alpha \right] \delta(t) \] \quad (3.2a)

\[ \mu_\alpha(t) = -\left[ \mu_\infty \theta \lim_{\alpha \to 0} \phi(\alpha) \left[ 1 - e^{-\alpha t} \right] d\alpha \right] \delta(t) \] \quad (3.2b)

where \( \theta < 0 \) is a constant having the dimension of time and the limiting process is defined after the function \( \phi(\alpha) \) is chosen.

Eringen [20] does not go far enough to suggest the form of \( \phi(\alpha) \). In this research, various forms were tried but the end result was that they were mathematically cumbersome and were difficult in numerical implementation. Finally, it was decided to choose a \( \phi(\alpha) \) that satisfies two criteria: first, the spectrum function must be a monotonically decreasing function of time in the interval \([0, \infty]\) so that physical properties, such as the modulus (\( E(t) \)) can be modeled correctly. Another requirement that is considered is that under some limiting condition, the transforms of \( \lambda_\alpha(t) \) and \( \mu_\alpha(t) \) become constants. These constants can be taken as the values obtained through Eq. (3.2). This requirement is important to fulfill for materials satisfying the K-V model. The only
function which when multiplied by \( 1 - e^{-\alpha t} \) and integrated over the range \( 0 \leq \alpha \leq \infty \) that can give constant values \( \lambda_{vc} \) and \( \mu_{vc} \) of the terms in square brackets in Eq. (3.2) is the Dirac delta function.

There are many choices for the spectrum function, \( \phi(\alpha) \). Exponential functions and the error function were tried, but these yielded expressions for which the Laplace inversion became very difficult. The spectrum function that satisfies the criteria of being non-negative, bounded, monotonic, and able to produce constant values in a particular limiting condition is selected. The proposed form chosen for this research and quoted in [24] is

\[
\varphi(\alpha) = \frac{r}{\pi(k^2 + r^2 \alpha^2)} \tag{3.3}
\]

where \( r \) and \( k \) are the temperature dependent parameters that are assumed to vary with the material. From Figure 3.1, it is seen that when \( \varphi(\alpha) \) is plotted, it produces a monotonic function that increases from \( \alpha = -\infty \) to \( \alpha = 0 \) and decreases from \( \alpha = 0 \) to \( \alpha = \infty \). For this study, only the positive half of the spectrum function is used.
In Eq. (3.3) $k$ is non-dimensional and $r$ has dimension of 1/sec. This choice of $\phi(\alpha)$ produces a constant value in the limiting process. It is immediately seen that as $r$ tends to infinity,

$$
\lim_{r \to \infty} \phi(\alpha) = \frac{1}{k^2} \frac{\delta\left(\frac{\alpha}{k}\right)}{k} = \frac{1}{k} \delta(\alpha), \quad k > 0
$$

(3.4)

where $\delta(\alpha)$ is the Dirac delta function, and this is the limiting process alluded to earlier.

On using (3.4) in (3.2) and (Eq. (A.4) in Appendix A), the following is obtained:

$$
\lim_{r \to \infty} \lambda_v(t) = -\frac{\lambda_v \theta}{2k} \delta(t)
$$

$$
\lim_{r \to \infty} \mu_v(t) = -\frac{\mu_v \theta}{2k} \delta(t)
$$

(3.5)

when $\theta < 0$. These constant values are the values $\lambda_{vc}$ and $\mu_{vc}$ which appear in the K-V model. As can be seen, both $\lambda_v(t)$ and $\mu_v(t)$ are then of the form of Eq. (2.30). Here $\theta$ is a
constant dependent on the material. It must be emphasized here that the forms given in Eq. (3.5) are purely demonstrative and no attempt has been made to establish the constants $k$ and $\theta$ for the results of this research. These values simply demonstrate that under the suggested \textit{limiting} condition the adopted function $\varphi(\alpha)$ in Eq. (3.3) yields constant values. For general viscoelastic materials, $\varphi(\alpha)$ is taken as

$$\varphi(\alpha) = \frac{n}{\pi(1 + n_o^2 \alpha^2)}$$

(3.6)

where the parameters are $n = \frac{r}{k^2}$ and $n_o = \frac{r}{k}$, and now there is no limiting process involved in the definition of the function in Eq. (3.6). Here $\varphi$ is regarded as a general spectrum function which is continuous and continuously differentiable. The validity of this $\varphi$ and the values of the constants will be established through comparison with the experimental data. Note that the dimension of both $n$ and $n_o$ is 1/sec.

\textbf{3.2 Evaluation of the Viscoelastic Parameters}

This section is concerned with the development of the memory functions $\lambda_v(t)$ and $\mu_v(t)$ and their Laplace transforms, $\lambda(\zeta)$ and $\mu(\zeta)$. The necessity to form the latter is well illustrated from the flow charts in Appendix B.

\textbf{3.2.1 Determination of $\lambda_v(t)$ and $\mu_v(t)$}

Since both $\lambda_v(t)$ and $\mu_v(t)$ are expressed in a similar fashion as shown by Eq. (3.1), only $\lambda_v(t)$ will be developed. First, the spectrum function in Eq. (3.6) is substituted in Eq. (3.1):
\[ \lambda_v(t) = -\lambda_e \int_0^\infty \frac{n}{\pi (1 + n_o^2 \alpha^2)} (1 - e^{-\alpha^2}) \, d\alpha \]  

(3.7)

This integral is now separated into two integrals:

\[ \lambda_v(t) = -\lambda_e \frac{n}{\pi} \left[ \int_0^\infty \frac{1}{(1 + n_o^2 \alpha^2)} \, d\alpha - \int_0^\infty \frac{1}{(1 + n_o^2 \alpha^2)} (e^{-\alpha^2}) \, d\alpha \right] \]  

(3.8)

The first integral is easily evaluated as

\[ \frac{n}{\pi} \int_0^\infty \frac{1}{(1 + n_o^2 \alpha^2)} \, d\alpha = \frac{n}{2n_o} \]  

(3.9)

After much rigorous effort and consultation of the integration tables, the second integral is found to be

\[ \frac{n}{\pi} \int_0^\infty \frac{e^{-\alpha^2}}{(1 + n_o^2 \alpha^2)} \, d\alpha = -\frac{n}{\pi n_o} \left[ \text{ci}(n_o t) \sin(n_o t) + \text{si}(n_o t) \cos(n_o t) \right] \]  

(3.10)

where \text{ci} and \text{si} are the cosine and sine integral functions, respectively, as defined in Appendix A.

Now, \( \lambda_v(t) \) can be formulated by substituting Eq. (3.9) and Eq. (3.10) into Eq. (3.8), resulting in

\[ \lambda_v(t) = -\lambda_e \left\{ \frac{n}{2n_o} + \frac{n}{\pi n_o} \left[ \text{ci}(n_o t) \sin(n_o t) + \text{si}(n_o t) \cos(n_o t) \right] \right\} \]  

(3.11)

Letting

\[ m = \frac{n}{n_o} \]  

(3.12)

and using the definitions in Appendix A, \( \lambda_v(t) \) is expressed as:
\[
\lambda_v(t) = -\frac{\lambda_c m}{2} \left\{ 1 - \frac{2}{\pi} \int_{z}^{\infty} \frac{\sin(z - n_o t)}{z} \, dz \right\} \quad (3.13a)
\]

Similarly, \( \mu_v(t) \) is expressed as
\[
\mu_v(t) = -\frac{\mu_c m}{2} \left\{ 1 - \frac{2}{\pi} \int_{z}^{\infty} \frac{\sin(z - n_o t)}{z} \, dz \right\} \quad (3.13b)
\]

### 3.2.2 Formulation of \( \lambda(\zeta) \) and \( \mu(\zeta) \)

Once the functional forms of \( \lambda_v(t) \) and \( \mu_v(t) \) have been established, the Laplace transform of the functions must be obtained. The Laplace transform of Eq. (3.13) is found to be [25]:
\[
\tilde{\lambda}_v(\zeta) = \mathcal{L} [\lambda_v(t)] = -\lambda_c \left[ \frac{m}{2\zeta} + \frac{m}{\pi n_o} \left( -\frac{\pi}{2} \left( \frac{\zeta}{n_o} \right)^2 + \ln \left( \frac{\zeta}{n_o} \right) \right) \right] \quad (3.14)
\]

To determine \( \lambda(\zeta) \), Eq. (2.22) is restated:
\[
\lambda(\zeta) = \lambda_c + \zeta \tilde{\lambda}(\zeta') \quad (3.15)
\]

Substituting Eq. (3.14) into Eq. (3.15), \( \lambda(\zeta) \) is obtained:
\[
\lambda(\zeta) = \lambda_c - \frac{m}{2} \frac{m}{\pi n_o} \left( \frac{\zeta \ln \left( \frac{\zeta}{n_o} \right)}{n_o} - \frac{\pi}{2} \left( \frac{\zeta}{n_o} \right)^2 + 1 \right) \quad (3.16)
\]
Letting,
\[
\bar{f}\left(\frac{\zeta}{n_o}\right) = \frac{m}{\pi} \left( \frac{\zeta}{n_o} \ln \frac{\zeta}{n_o} - \frac{\pi}{2} \left( \frac{\zeta}{n_o}\right)^2 \right)
\]

Equation (3.16) can now be expressed as
\[
\lambda(\zeta) = \lambda_c \left[ 1 - \frac{m}{2} \right] \bar{f}\left(\frac{\zeta}{n_o}\right)
\]
(3.18a)

Similarly,
\[
\mu(\zeta) = \mu_c \left[ 1 - \frac{m}{2} \right] \bar{f}\left(\frac{\zeta}{n_o}\right)
\]
(3.18b)

Up to this point, all developments have been for the isotropic, homogeneous class of materials. For a two-layered composite laminate with each layer made of a different material, the following forms are taken
\[
\lambda_1(\zeta) = \lambda_{c1} H_1(\zeta) \quad , \quad \mu_1(\zeta) = \mu_{c1} H_1(\zeta)
\]
\[
\lambda_2(\zeta) = \lambda_{c2} H_2(\zeta) \quad , \quad \mu_2(\zeta) = \mu_{c2} H_2(\zeta)
\]
(3.19)

where the subscript 1 and 2 signify the two materials, and analogous to Eq. (3.18),
\[
H_1(\zeta) = \left(1 - \frac{m_1}{2}\right) \bar{f}\left(\frac{\zeta}{n_{o1}}\right)
\]
\[
H_2(\zeta) = \left(1 - \frac{m_2}{2}\right) \bar{f}\left(\frac{\zeta}{n_{o2}}\right)
\]
(3.20)

These functions are then used to determine the viscoelastic parameters as functions of time. As can be seen, for a medium composed of two materials, there are four constants to be determined: \(m_1, n_{o1}, m_2, n_{o2}\) instead of two constants (m and \(n_o\)) for the isotropic homogeneous case. These constants are used in the development of the Lame’ functions
\( \lambda_v(t) \) and \( \mu_v(t) \) which are solely functions of time. The constants, \( m \) and \( n_0 \), are not functions of layer thickness or volume – they are material property constants that may be temperature dependent. Figure B.2 in Appendix B shows the overall procedure in the determination of \( \lambda_v(t) \) and \( \lambda(\zeta) \).
CHAPTER IV
THE APPLICATION OF THE PRINCIPLE OF CORRESPONDENCE

The premise of the Principle of Correspondence has already been set in Section 2.4. In this chapter, the principle will be applied to three different test cases: tensile, flexure - four point bend, and flexure – cantilevered. The first two cases are considered because of the availability of the experimental data [21,22] for comparison with the corresponding theoretical solutions. This comparison brings out the viscoelastic parameters and the functional forms of the viscoelastic functions and their variations with time. Also, for demonstrative purposes, the viscoelastic behavior of the cantilevered beam for both the homogeneous as well as the non-homogeneous layered solid is determined by using a Kelvin-Voigt model.

Before beginning, it is important to summarize the similarity of forms of the elastic and the corresponding viscoelastic mechanical parameters that will be obtained through the use of the Correspondence Principle. In Table 4.1, it is important to observe the forms of each elastic term and compare it to its corresponding viscoelastic term. The appearance of the Laplace parameter $\zeta$ on the right hand side comes from taking the Laplace transform of the constant elastic terms. This table is helpful when applying the Correspondence Principle to any elastic equation.
**Table 4.1: Elastic-Viscoelastic Correspondence of Properties**

<table>
<thead>
<tr>
<th>Description</th>
<th>Elastic</th>
<th>Viscoelastic Analog</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus</td>
<td>( E_e = \frac{\mu_e(3\lambda_e + 2\mu_e)}{\lambda_e + \mu_e} )</td>
<td>( \zeta \overline{E}(\zeta) = \frac{\mu(\zeta)[3\lambda(\zeta) + 2\mu(\zeta)]}{\lambda(\zeta) + \mu(\zeta)} )</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>( \nu_e = \frac{\lambda_e}{2(\lambda_e + \mu_e)} )</td>
<td>( \zeta \overline{\nu}(\zeta) = \frac{\lambda(\zeta)}{2[\lambda(\zeta) + \mu(\zeta)]} )</td>
</tr>
<tr>
<td>Bulk Modulus</td>
<td>( K_e = \lambda_e + \frac{2}{3}\mu_e )</td>
<td>( \zeta \overline{K}(\zeta) = \lambda(\zeta) + \frac{2}{3}\mu(\zeta) )</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>( G_e = \mu_e )</td>
<td>( \zeta \overline{G}(\zeta) = \mu(\zeta) )</td>
</tr>
<tr>
<td>Bulk Compliance</td>
<td>( B_e = \frac{1}{K_e} = \frac{1}{\lambda_e + \frac{2}{3}\mu_e} )</td>
<td>( \zeta \overline{B}(\zeta) = \frac{1}{\lambda(\zeta) + \frac{2}{3}\mu(\zeta)} = \frac{1}{\zeta \overline{K}(\zeta)} )</td>
</tr>
<tr>
<td>Shear Compliance</td>
<td>( J_e = \frac{1}{G_e} = \frac{1}{\mu_e} )</td>
<td>( \zeta \overline{J}(\zeta) = \frac{1}{\mu(\zeta)} = \frac{1}{\zeta \overline{G}(\zeta)} )</td>
</tr>
</tbody>
</table>

### 4.1 Uniaxial Creep Testing

Quasi-static testing is conducted at various temperatures to obtain the elastic properties as functions of temperature. Creep tests are implemented in tension or compression at a constant load inside a temperature controlled chamber. The extension or strain is measured as a function of time. In this section, equations are developed to evaluate the rate dependent properties such as strain, modulus, compliance and Poisson’s ratio.
4.1.1 Determination of Strain, $e(t)$

In uniaxial testing (compression or tension), only one stress component exists in the direction of load. Hooke’s Law for linearly elastic isotropic materials in uniaxial tension is

$$e_{11} = \frac{T_{11}}{E_e}$$

(4.1)

where $E_e$ has been defined in terms of Lame’ constants in Eq. (2.3) and in Table 4.1. Substituting this expression into Eq. (4.1) gives

$$e_{11} = \frac{\lambda_e + \mu_e}{\mu_e (3\lambda_e + 2\mu_e)} T_{11}$$

(4.2)

The Principle of Correspondence states that if the linear elastic solution exists, then the corresponding linear viscoelastic solution can be found by replacing the parameters with corresponding substitutions. The Correspondence Principle can now be applied to Eq. (4.2) and the strain as a function of the Laplace parameter $\zeta$ is obtained:

$$\bar{e}_{11} = \frac{\lambda(\zeta) + \mu(\zeta)}{\mu(\zeta) [3\lambda(\zeta) + 2\mu(\zeta)]} \frac{T_{11}}{\zeta}$$

(4.3)

The transformed equation, Eq. (4.3), can also be obtained by another method. Using the Boltzman-Volterra equation, Eq. (2.15), for the special case when the stress in only one direction exists, the following is obtained.

$$T_{11} = \int_0^t K(t - \tau) \frac{\partial [e_{11}(\tau)]}{\partial \tau} d\tau + 2 \int_0^t G(t - \tau) \frac{\partial [e_{11}(\tau)]}{\partial \tau} d\tau$$

(4.4a)
\[ T_{22} = \int_0^1 K(t - \tau) \frac{\partial [e_n(\tau)]}{\partial \tau} d\tau + 2 \int_0^1 G(t - \tau) \frac{\partial [\bar{e}_{22}(\tau)]}{\partial \tau} d\tau = 0 \] (4.4b)

\[ T_{33} = \int_0^1 K(t - \tau) \frac{\partial [e_n(\tau)]}{\partial \tau} d\tau + 2 \int_0^1 G(t - \tau) \frac{\partial [\bar{e}_{33}(\tau)]}{\partial \tau} d\tau = 0 \] (4.4c)

where

\[ \bar{e}_{ij} = e_{ij} - \frac{1}{3} e_n \delta_{ij} \]

The Laplace transforms of Eqs. (4.4) can now be taken by using the convolution theorem.

\[ \bar{T}_{11}(\zeta) = \zeta \bar{e}_n(\zeta) \left[ \bar{K}(\zeta) - \frac{2}{3} \bar{G}(\zeta) \right] + 2 \zeta \bar{G}(\zeta) \bar{e}_{11}(\zeta) \] (4.5a)

\[ \bar{T}_{22}(\zeta) = \zeta \bar{e}_n(\zeta) \left[ \bar{K}(\zeta) - \frac{2}{3} \bar{G}(\zeta) \right] + 2 \zeta \bar{G}(\zeta) \bar{e}_{22}(\zeta) = 0 \] (4.5b)

\[ \bar{T}_{33}(\zeta) = \zeta \bar{e}_n(\zeta) \left[ \bar{K}(\zeta) - \frac{2}{3} \bar{G}(\zeta) \right] + 2 \zeta \bar{G}(\zeta) \bar{e}_{33}(\zeta) = 0 \] (4.5c)

Solving Eqs. (4.5) simultaneously results in the following expression for the strain:

\[ \bar{e}_{i1}(\zeta) = \frac{\bar{T}_{11}(\zeta)}{\zeta \bar{E}(\zeta)} \] (4.6)

Using Eq. (2.26) for \( \zeta \bar{E}(\zeta) \) in Eq. (4.6), Equation (4.3) is recovered. The viscoelastic memory functions, \( \lambda(\zeta) \) and \( \mu(\zeta) \), have already been developed in Section 3.2, Eq. (3.18) by using the spectrum function \( \phi(\alpha) \). Using these results in Eq. (4.3) results in an expression for the strain as a function of the Laplace parameter \( \zeta \).

\[ \bar{e}_{i1}(\zeta) \left[ \left(1 - \frac{m}{2} \right) - \bar{f} \left( \frac{\zeta}{n_o} \right) \right] = \frac{\bar{T}_{11}(\zeta)}{E_c} \] (4.7)

The inverse Laplace of each term is now taken, resulting in
\[
(1 - \frac{m}{2}) e_{11}(t) - \mathcal{L}^{-1} \left[ \bar{e}_{11}(\zeta) \tilde{f}\left(\frac{\zeta}{n_o}\right) \right] = \frac{T_{11}(t)}{E_c}
\] (4.8)

Using the Laplace tables, [25]
\[
\mathcal{L}^{-1} \left[ \tilde{f}\left(\frac{\zeta}{n_o}\right) \right] = \frac{m n_o}{\pi} \left[ \cos(n_o t) \text{ci}(n_o t) - \sin(n_o t) \text{si}(n_o t) - \frac{\pi}{2} \delta(n_o t) \right]
\] (4.9)

The convolution theorem is then used to perform the inverse in Eq. (4.8):
\[
\left(1 - \frac{m}{2}\right) e_{11}(t) - \frac{m n_o}{\pi} \int_0^t e_{11}(t-\tau) \left[ \cos(n_o \tau) \text{ci}(n_o \tau) - \sin(n_o \tau) \text{si}(n_o \tau) - \frac{\pi}{2} \delta(n_o \tau) \right] d\tau = \frac{T_{11}(t)}{E_c}
\] (4.10)

Using the properties of the Dirac delta function, the strain as a function of time is obtained.
\[
e_{11}(t) = \frac{m n_o}{\pi} \int_0^t e_{11}(t-\tau) \left[ \cos(n_o \tau) \text{ci}(n_o \tau) - \sin(n_o \tau) \text{si}(n_o \tau) \right] d\tau = \frac{T_{11}(t)}{E_c}
\] (4.11)

Using the identities given in Appendix A, the bracketed term in the integral in Eq. (4.11) can also be expressed as
\[
\psi(t) = \cos(n_o t) \text{ci}(n_o t) - \sin(n_o t) \text{si}(n_o t)
\]
\[
= \cos(n_o t) \int_{n_o t}^{\infty} \frac{\cos z}{z} dz - \sin(n_o t) \int_{n_o t}^{\infty} \frac{\sin z}{z} dz
\]
\[
= \int_{n_o t}^{\infty} \frac{\cos(n_o t) \cos z - \sin(n_o t) \sin z}{z} \ dz
\]

Therefore, \(\psi(\tau)\) is written as
\[
\psi(t) = \int_{n_o t}^{\infty} \frac{\cos(z - n_o t)}{z} \ dz
\] (4.12)

Now, the strain as a function of time is
Using Eq. (3.12) and knowing that the stress is constant for tension creep experiments, i.e.,
\[ T_{11}(t) = \sigma = \text{constant} \]
the strain as a function of time is expressed in Eq. (4.14).
\[
e_{11}(t) - \frac{n}{\pi} \int_{0}^{t} e_{11}(t-\tau) \psi(\tau) d\tau = \frac{\sigma}{E_c} \tag{4.14}
\]
Equation (4.14) is a Volterra integral equation and it can be solved by the method of iteration as
\[
e_{11}^{(p+1)}(t) = \frac{\sigma}{E_c} + \frac{n}{\pi} \int_{0}^{t} e_{11}^{(p)}(t-\tau) \psi(\tau) d\tau \quad p = 1, 2, 3, ... \tag{4.15}
\]
where \( e_{11}^{(p)} \) is the \( p \)th approximation and \( e_{11}^{(p+1)} \) is the \((p+1)\)st approximation. Section 4.5 includes an algorithm which provides a procedure that can be used to evaluate such equations. The full procedure to obtain strain as a function of time is depicted by the flow chart in Figure B.3, Appendix B.

4.1.2 Determination of Young’s Modulus, \( E(t) \)

The memory functions have been stated previously in Eq. (2.24) and Eq. (2.25).

In the case of linear elasticity, these expressions reduce to the Laplace transform of Young’s Modulus and Poisson’s ratio, that is
\[
\bar{E}(\zeta) = \frac{E_c}{\zeta}, \quad \bar{\nu}_c(\zeta) = \frac{\nu_c}{\zeta}
\]
Once the Laplace transforms of the moduli in Eqs.(2.12) are taken and substituted into Eq. (2.24), \( \bar{E}(\zeta) \) can be expressed as

\[
\bar{E}(\zeta) = \left( \frac{1}{\zeta} \right) \frac{\mu(\zeta)(3\lambda(\zeta) + 2\mu(\zeta))}{\lambda(\zeta) + \mu(\zeta)}
\] (4.16)

The previously obtained expression for \( \lambda(\zeta) \) and \( \mu(\zeta) \) from Eq. (3.18) are now substituted into Eq. (4.16) to obtain

\[
\zeta \bar{E}(\zeta) = E_c \left[ \left( 1 - \frac{m}{2} \right) - \tilde{f} \left( \frac{\zeta}{n_o} \right) \right]
\] (4.17)

where \( \tilde{f} \left( \frac{\zeta}{n_o} \right) \) has been defined in Eq. (3.17). Solving for \( \bar{E}(\zeta) \) from Eq. (4.17), the following form is obtained:

\[
\bar{E}(\zeta) = E_c \left[ \left( 1 - \frac{m}{2} \right) - \frac{m}{\pi n_o} \left( \ln \frac{\zeta}{n_o} - \frac{\pi}{2} \frac{\zeta}{n_o} + 1 + \frac{\pi}{2} \left( \frac{\zeta}{n_o} \right)^2 \right) \right]
\] (4.18)

The modulus as a function of time, \( E(t) \), can now be obtained by taking the inverse transform of Eq. (4.18).

\[
E(t) = E_c \left[ \left( 1 - \frac{m}{2} \right) + \frac{m}{\pi} \int_{n_o t}^{\infty} \sin \left( z - n_o t \right) \frac{dz}{z} \right]
\] (4.19)

The selection of the constants \( n \) and \( n_o \) is discussed in Section 4.5. Once these choices have been made, Eq. (4.19) can be easily evaluated for modulus as a function of time. No approximation need be made when performing this calculation. The procedure for obtaining \( E(t) \) is shown in the flow chart in Figure B.4, Appendix B.
4.1.3 Determination of Poisson’s Ratio

To determine the Poisson’s ratio as a function of time, Eq. (2.25) is restated:

\[
\nu(\zeta) = \frac{3K(\zeta) - 2G(\zeta)}{6K(\zeta) + 2G(\zeta)} \cdot \frac{1}{\zeta}
\]

(4.20)

From Lakes [26], an explicit equation for the Poisson’s ratio as a function of time and which is consistent with Eq. (4.20) is

\[
\nu(t) = \frac{1}{2} - \frac{1}{6} \int_0^t E(t - \tau) \frac{dB(\tau)}{d\tau} d\tau
\]

(4.21)

where \(B(t)\) is the bulk creep compliance. Lakes [26] states that for a polymeric material the ratio of the variation between the shear relaxation modulus \(G(t)\) and the bulk modulus \(K(t)\) is as much as 500 over the complete time range. Due to this, the bulk modulus is approximated as a constant. Another way to obtain a form of \(\nu(\zeta)\) is first to note the elastic definition of the bulk modulus in terms of the elastic Young’s modulus and Poisson’s ratio:

\[
K_e = \frac{E_e}{3(1 - 2\nu_e)}
\]

(4.22)

Solving for the Poisson’s ratio, \(\nu_e\) is expressed as

\[
\nu_e = \frac{1}{2} - \frac{1}{6} \frac{E_e}{K_e}
\]

Replacing \(\nu_e\) by \(\nu(\zeta)\), \(E_e\) by \(E(\zeta)\), \(K_e\) by \(K(\zeta)\), the following is obtained.

\[
\nu(\zeta) = \frac{1}{2} - \frac{1}{6} \frac{E(\zeta)}{K(\zeta)}
\]

This type of replacement is an application of the Principle of Correspondence. Using
$\tilde{E}(\zeta)$ and $\tilde{K}(\zeta)$ from Table 4.1, the following form is obtained

$$\zeta v(\zeta) = \frac{\lambda(\zeta)}{2[\lambda(\zeta) + \mu(\zeta)]}$$

Therefore, a close approximation to Eq. (4.21) is expressed as

$$v(t) = \frac{1}{2} - \frac{1}{6} \frac{E(t)}{K_e}$$  \hspace{1cm} (4.23)

which in effect means that for the purpose of calculating the Poisson’s ratio in Eq. (4.23), the bulk compliance $B(t)$ in Eq. (4.21) is taken as a constant and $K_e$ is the elastic bulk modulus defined in Eq. (4.22). Using Eq. (4.19), $v(t)$ is obtained.

$$v(t) = \frac{1}{2} - \frac{3\mu_e}{6(\lambda_e + \mu_e)} \left[ \left( 1 - \frac{m}{2} \right) + \frac{m}{\pi} \int_{n_{st}}^{\infty} \frac{\sin(z - n_{st}l)}{z} \, dz \right]$$  \hspace{1cm} (4.24)

The procedure for obtaining $v(t)$ is shown in the flow chart in Figure B.5, Appendix B.

For layered media, the analysis would also begin from Eq. (4.23), but instead of using $E(t)$ and $K_e$, an effective modulus and bulk modulus must be formed that are in terms of the Lame’ parameters of the materials involved. The reader is referred to Section 4.2 regarding the formation of the effective flexural modulus.

### 4.1.4 Determination of the Creep Bulk Compliance $B(t)$

To determine the creep bulk compliance $B(t)$, Eq. (2.21) is first written as shown in Eq. (4.25).

$$\overline{B}(\zeta) = \frac{1}{\zeta^2 \overline{K}(\zeta)}$$  \hspace{1cm} (4.25)

The expression for the bulk relaxation modulus in terms of the Laplace parameter $\zeta$ must be obtained by taking the Laplace transform of Eq. (2.12a) and can be written as
\[ \mathcal{K}(\zeta) = \frac{1}{\zeta} \left[ \lambda(\zeta) + \frac{2}{3} \mu(\zeta) \right] \]  
\hspace{10cm} (4.26)

Substituting Eq. (3.18) into Eq. (4.26) and substituting the result into Eq. (4.25) gives

\[ B(\zeta) = \left( \frac{3}{\zeta(3\lambda_e + 2\mu_e)} \right) \left( \frac{1-m}{2} - \frac{\zeta}{n_o} \right) \]  
\hspace{10cm} (4.27)

Using the convolution theorem and Eq. (A.5) in Appendix A, the bulk compliance as a function of time is obtained.

\[ B(t) - \frac{mn_o}{\pi} \int_0^t B(t-\tau) \psi(\tau) d\tau = \frac{3}{3\lambda_e + 2\mu_e} \]  
\hspace{10cm} (4.28)

where \( \psi(t) \) has been previously defined in Eq. (4.12). Again, the resulting equation is a Volterra integral equation that can be solved as shown in Eq. (4.15). The procedure for obtaining \( B(t) \) is shown in the flow chart in Figure B.6, Appendix B.

4.1.5 Creep Compliances

At this time it is necessary to differentiate between the creep bulk compliance \( B(t) \) and the more commonly used creep compliance \( S(t) \). The time dependent creep compliance \( S(t) \) is the property that is generally determined via experimental testing.

Letting the creep bulk compliance be stated as

\[ B(t) = \frac{3\mu_e}{\lambda_e + \mu_e} S(t) \]  
\hspace{10cm} (4.29)
and substituting Eq. (4.29) in Eq. (4.28) gives

$$S(t) - \frac{n}{\pi} \int_0^t S(t-\tau) \psi(\tau) \, d\tau = \frac{1}{E_e}$$

(4.30)

where $E_e$ is the Young’s modulus of linear elasticity. The creep compliance $S(t)$ is defined as

$$S(t) = \frac{e_{11}(t)}{\sigma}$$

(4.31)

Using Eq. (4.31) in Eq. (4.30) results in the following equation,

$$e_{11}(t) - \frac{n}{\pi} \int_0^t e_{11}(t - \tau) \psi(\tau) \, d\tau = \frac{\sigma}{E_e}$$

(4.32)

which is exactly Eq. (4.14). From this, it is concluded that due to the definition in Eq. (4.31), there is no need to evaluate the integral in Eq. (4.30) for $S(t)$. Once the strain as a function of time is known, the time dependent creep compliance can be determined simply by dividing it by the constant stress value, as shown in Eq. (4.31).

It is also important to establish the relationship between the creep compliance $S(t)$, the creep bulk compliance $B(t)$, and the shear compliance $J(t)$. To this end, the linear viscoelastic constitutive equation in Eq. (2.19) is restated:

$$e_{ij} = \frac{1}{9} \delta_{ij} \int_0^t B(t - \tau) \frac{\partial T_{rr}}{\partial \tau} \, d\tau + \frac{1}{2} \int_0^t J(t - \tau) \frac{\partial T_{ij}}{\partial \tau} \, d\tau$$

(4.33)

In the two-dimensional case, i.e., $i, j = 1, 2$, Eq. (4.33) is written as

$$e_{11}(t) = \int_0^t \left\{ \frac{1}{9} B(t - \tau) + \frac{1}{3} J(t - \tau) \right\} \frac{dT_{11}}{d\tau} + \left\{ \frac{1}{9} B(t - \tau) - \frac{1}{6} J(t - \tau) \right\} \frac{dT_{22}}{d\tau} \, d\tau$$

(4.34)
From Gibson [4], in terms of creep compliances, $S_{ij}$, the Boltzmann superposition integral for the plane stress case for the strain in the longitudinal direction is written as

$$e_{11}(t) = \int_0^t \left[ S_{11}(t - \tau) \frac{dT_{11}}{d\tau} + S_{12}(t - \tau) \frac{dT_{22}}{d\tau} \right] d\tau$$  \hspace{1cm} (4.35)

Comparing Eq. (4.34) and Eq. (4.35), the relationship between the compliances is found to be:

$$S_{11}(t) = \frac{1}{9} B(t) + \frac{1}{3} J(t)$$  \hspace{1cm} (4.36)

where $S_{11}(t)$ is equal to $S(t)$, the creep compliance. As stated before, in creep experiments, the creep compliance is determined by Eq. (4.31) and this includes the effects of both the bulk shear compliance and the shear compliance. It must also be emphasized that, in general, due to varying time patterns [3]

$$S(t) \neq \frac{1}{E(t)}$$

where $E(t)$ is the relaxation modulus. These properties are related to each other by their Laplace transforms which are analogous to the bulk properties given in Eq. (4.25).

$$\overline{S}(\zeta) = \frac{1}{\zeta^2 \overline{E}(\zeta)}$$  \hspace{1cm} (4.37)

This is easily shown by taking the Laplace transform of Eq. (4.36).

$$\overline{S}(\zeta) = \frac{1}{9} \overline{B}(\zeta) + \frac{1}{3} \overline{J}(\zeta)$$  \hspace{1cm} (4.38)

Substituting the expressions for $\overline{B}(\zeta)$ and $\overline{J}(\zeta)$ from Eq. (2.21) into Eq. (4.38) yields

$$\overline{S}(\zeta) = \frac{1}{\zeta^2} \frac{\overline{G}(\zeta) + 3\overline{K}(\zeta)}{9\overline{K}(\zeta)\overline{G}(\zeta)}$$  \hspace{1cm} (4.39)
Comparing this expression with Eq. (2.24), it is seen that Eq. (4.39) and Eq. (4.37) are identical.

4.2 Four-Point Flexure Creep Testing

A flexure test is quite often employed for determining the behavior of various material systems by subjecting the test piece to either a three-point bend test (central loading) or a four-point bend test. Figure 4.1 shows the schematic of the four-point flexure test used in this study.

Figure 4.1: Beam under four-point loading.

The data was available for beams subjected to a four-point bend test at several temperatures in flexure creep [22]. The beams were either laminated monolithic beams, composed entirely of a fiberglass layup, or sandwich beams, with composite face sheets and balsa core. Table 4.2 shows the geometric characteristics as well as the beams’ layup patterns.
Table 4.2: Flexure Beam Dimensions and Layup Sequence

<table>
<thead>
<tr>
<th>Beam Type</th>
<th>Span Length - m (in)</th>
<th>Width – m (in)</th>
<th>Thickness - m (in)</th>
<th>Layup Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid</td>
<td>1.524 (60)</td>
<td>0.1524 (6)</td>
<td>0.0254 (1)</td>
<td>[0/90/45/-45]₄S balanced weave</td>
</tr>
<tr>
<td>Sandwich</td>
<td>1.524 (60)</td>
<td>0.1524 (6)</td>
<td>0.0635 (2.5)</td>
<td>[0/90/45/-45/core]₅S balanced weave</td>
</tr>
</tbody>
</table>

For the analysis in this study, the solid laminate beam is treated as a homogeneous, isotropic beam due to the large numbers of ply groups in its layup. This approximation is based on Figure 4.2 from Reference [27], which depicts the relationship between the flexural modulus components and \( p \), the number of ply groups, for a T300/5208 laminate.

![Figure 4.2: Flexural modulus components as ply group \( p \) increases [27].](image)

Ply groups are defined to be groups of plies of the same ply orientation and material.

Figure 4.2 shows that as the number of ply group \( p \) increases, the flexural modulus
components converge to that of a quasi-homogeneous laminate. As seen in Table 4.2, the quasi-isotropic layup of the fiberglass laminate results in thirty-two ply groups. Therefore, the solid laminate can be regarded as a quasi-homogeneous solid and the available effective laminate properties are used in the analysis. The sandwich beam is treated as a laminated structure, having composite face sheets and a balsa core.

It is recognized that shear effects can play a key role in deformation analysis in composites. These effects are mainly a function of two ratios: span-to-thickness ratio and the moduli ratio. It is determined that shear effects play an insignificant role for the monolithic beam in four point flexure, and for this study, shear deformation was not included in the analysis for either the monolithic or sandwich beam. However, shear effects are discussed in Section 4.4 and implementation of these effects is demonstrated in Section 4.4.2.

4.2.1 Isotropic Beam

For a beam in four point flexure as shown in Figure 4.1, the standard mechanics of material solution gives the center deflection as

$$y_c = \frac{Pa}{12E_cI} \left( -\frac{3}{4}L^2 + a^2 \right)$$  \hspace{1cm} (4.40)

If

$$a = \frac{L}{3} \quad \text{and} \quad \beta = -\frac{23}{1296} \frac{L^3}{I},$$

the center deflection can be written as

$$y_c = \frac{\beta}{E_c} P$$  \hspace{1cm} (4.41)
For a homogeneous, isotropic beam, \( E_e \) is the Young’s Modulus as given in Eq. (2.3).

Applying the Principle of Correspondence to Eq. (4.41), the transform of the center deflection is written as

\[
\bar{y}_c(\zeta) = \beta \left[ \frac{\lambda(\zeta) + \mu(\zeta)}{\mu(\zeta)[3\lambda(\zeta) + 2\mu(\zeta)]} \right] \bar{P}(\zeta) \tag{4.42}
\]

where \( \lambda(\zeta) \) and \( \mu(\zeta) \) have been defined in Eq. (3.18). The term in the square brackets is defined in Eq. (4.16) and Eq. (4.17), and Eq. (4.42) is written as

\[
\bar{y}_c(\zeta) = \beta \frac{1}{E_e \left[ \left( 1 - \frac{m}{2} \right) - \bar{f} \left( \frac{\zeta}{n_o} \right) \right]} \bar{P}(\zeta) \tag{4.43}
\]

To obtain the center deflection as a function of time, the Laplace inverse of Eq. (4.43) is taken:

\[
y_c(t) - \frac{n}{\pi} \int_0^t y_c(t - \tau) \psi(\tau) d\tau = \frac{\beta}{E_e} P(t) \tag{4.44}
\]

The resulting Volterra integral equation can be solved by the method of iteration as shown in Eq. (4.15). The procedure for obtaining \( y_c(t) \) is shown in the flow chart in Figure B.7, Appendix B.

### 4.2.2 Sandwich Beam

Since sandwich beams, composed of a balsa core and composite face sheets, are regarded as layered composite structures, the flexural modulus is now dependent on the ply stacking sequence. The face sheets and the balsa core are considered as two quasi-homogeneous materials and since their effective properties are available, Gibson’s equation [4] for the effective flexural modulus for a layered media is used. For an even
number of layers, the effective flexural modulus of the sandwich beam can be determined from

\[ E_f = \frac{8}{N^3} \sum_{j=1}^{N/2} (E_x)_j (3j^2 - 3j + 1) \]  

(4.45)

where \( N \) is equal to the total number of plies, including the core thickness expressed in equivalent number of plies. The effective properties of the face sheets are used because the face sheets are thin relative to the core and these are considered to be quasi-homogeneous. Therefore, the strain distribution is considered to be uniform, and a smeared model can be used. Having a sandwich beam with a cross-section as shown in Figure 4.3, Eq. (4.45) can also be written in a generic form as

\[ E_f = \kappa_0 [\kappa_1 E_1 + \kappa_2 E_2] \]  

(4.46)

where

\[ \kappa_0 = \frac{1}{(\kappa_1 + \kappa_2)} \]

and \( \kappa_1, \kappa_2 \) are constants that are determined by the order or sequence of the layup of the sandwich structure. The core and face sheets are designated by subscripts 1 and 2, respectively, with the face sheets and the core made of two separate materials (c.f. Section 5.2.2).
Using Lame’ constants, the effective modulus can also be written as:

$$E_f = \kappa_0 \left[ \frac{\mu_{e1} (3\lambda_{e1} + 2\mu_{e1})}{\lambda_{e1} + \mu_{e1}} + \frac{\mu_{e2} (3\lambda_{e2} + 2\mu_{e2})}{\lambda_{e2} + \mu_{e2}} \right]$$

(4.47)

From elastic theory, the deflection at the center of the beam can be expressed as

$$y_c = \frac{\gamma P}{E_f}$$

(4.48)

where $\gamma$ is a constant that is defined by the cross-sectional properties of the beam.

Substituting the effective modulus, Eq. (4.47) into Eq. (4.48) and applying the Principle of Correspondence gives

$$\bar{y}_c(\zeta) = \frac{\gamma \bar{P}(\zeta)}{\kappa_0} \left[ \frac{1}{\kappa_1 \frac{\mu_1(\zeta) (3\lambda_1(\zeta) + 2\mu_1(\zeta))}{\lambda_1(\zeta) + \mu_1(\zeta)} + \kappa_2 \frac{\mu_2(\zeta) (3\lambda_2(\zeta) + 2\mu_2(\zeta))}{\lambda_2(\zeta) + \mu_2(\zeta)}} \right]$$

(4.49)
where,

\[
\lambda_i(\zeta) = \lambda_e H_i(\zeta), \quad \mu_i(\zeta) = \mu_e H_i(\zeta)
\]

\[
\lambda_2(\zeta) = \lambda_{e2} H_2(\zeta), \quad \mu_2(\zeta) = \mu_{e2} H_2(\zeta)
\]

(4.50)

and analogous to Eqs. (3.18),

\[
H_1(\zeta) = \left(1 - \frac{m_i}{2}\right) - \int \frac{\zeta}{n_{o1}}
\]

\[
H_2(\zeta) = \left(1 - \frac{m_e}{2}\right) - \int \frac{\zeta}{n_{o2}}
\]

(4.51)

Therefore,

\[
\bar{y}_c(\zeta) = \frac{\gamma \bar{P}(\zeta)}{\kappa_0} \left[ \frac{1}{\kappa_1 E_{e1} H_1(\zeta) + \kappa_2 E_{e2} H_2(\zeta)} \right]
\]

(4.52)

Substituting Eq. (4.51) into Eq. (4.52) and performing the inverse Laplace and using the convolution theorem, the center deflection at \( t > 0 \) is determined to be

\[
y_c(t) = \frac{\alpha_1 n_1}{\pi} \int_0^t y_c(t - \tau) \psi_1(\tau) d\tau - \frac{\alpha_2 n_2}{\pi} \int_0^t y_c(t - \tau) \psi_2(\tau) d\tau = \alpha_3 P(t)
\]

(4.53)

where

\[
\alpha_1 = \frac{\kappa_1 E_{e1}}{\kappa_1 E_{e1} + \kappa_2 E_{e2}}
\]

\[
\alpha_2 = \frac{\kappa_2 E_{e2}}{\kappa_1 E_{e1} + \kappa_2 E_{e2}}
\]

\[
\alpha_3 = \frac{\gamma}{\kappa_0 (\kappa_1 E_{e1} + \kappa_2 E_{e2})}
\]

and

\[
\psi_1(t) = \int_{n_{o1}}^{\infty} \frac{\cos(z - n_{o1}t)}{z} dz, \quad \psi_2(t) = \int_{n_{o2}}^{\infty} \frac{\cos(z - n_{o2}t)}{z} dz
\]
\[ n_1 = m_1 n_{01}, \quad n_2 = m_2 n_{02} \]

text continues...
under the condition \( \frac{\partial y}{\partial x} = 0 \) at \( x = L \). All quantities are shown in Figure 4.4.

According to the Principle of Correspondence, Eq. (4.56) for a cantilever beam made of a homogeneous viscoelastic material can be expressed as

\[
y(x, \zeta) = \frac{P L^3}{3I} \frac{\lambda(\zeta) + \mu(\zeta)}{\mu(\zeta)(3\lambda(\zeta) + 2\mu(\zeta))} \left[ -\frac{1}{2} \left( 1 - \frac{x^3}{L^3} \right) + \frac{3}{2} \left( 1 - \frac{x}{L} \right) \right]
\] (4.57)

For the case of the Kelvin-Voigt model, the constitutive equation (Eq. 2.32) is

\[
T_{kl} = \left( \lambda_e + \lambda_{vc} \frac{\partial}{\partial t} \right) e_{e,k} \delta_{kl} + 2 \left( \mu_e + \mu_{vc} \frac{\partial}{\partial t} \right) e_{e,l}
\] (4.58)

It must be noted that \( \lambda_{vc} \) and \( \mu_{vc} \) are material constants. For a Kelvin-Voigt model, Eq. (2.33) is used with the quantities shown in Eq. (4.59).

\[
\eta_e = \frac{\lambda_e}{\mu_e}, \quad \eta_v = \frac{\lambda_{vc}}{\mu_{vc}}
\]

\[
\tau_1 = \frac{\mu_{vc}}{\mu_e}, \quad \tau_2 = \frac{3\lambda_{vc} + 2\mu_{vc}}{3\lambda_e + 2\mu_e}
\]

\[
r = \frac{\tau_2}{\tau_1}
\]

\[
z = \frac{1 + \eta_v}{1 + \eta_e} = \frac{1 + r(2 + 3\eta_e)}{3 + 3\eta_e}
\] (4.59)
In Eq. (4.59), \( \tau_1 \) and \( \tau_2 \) have dimension of time, whereas all other quantities are dimensionless. Substituting the quantities stated in Eq. (4.59) into Eq. (4.58), the inverse of the transform can be obtained. First, the load function \( P(t) \) is considered for two choices:

1: \( P(t) = P = \text{constant} \quad \rightarrow \quad \mathcal{F}[P(\zeta)] = \frac{P}{\zeta} \)

2: \( P(t) = \begin{cases} P = \text{const.} & 0 \leq t < t_m \\ 0, & t > t_m \end{cases} \quad \rightarrow \quad \mathcal{F}[P(\zeta)] = \frac{P(1 - e^{-\zeta t_m})}{\zeta} \)

For the first selection, the inverse transform is

\[
y(x, t) = kPF_1(x, t), \quad k = \frac{L^3}{3I} \quad (4.60)
\]

where

\[
F_1(x, t) = \frac{1}{(1 - r)} \left[ (1 - r) + r e^{-\zeta_2} - e^{-\zeta_1} + z \left( e^{-\zeta_1} - e^{-\zeta_2} \right) \right] \left[ -\frac{1}{2} \left( 1 - \frac{x^3}{L^3} \right) + \frac{3}{2} \left( 1 - \frac{x}{L} \right) \right] \quad (4.61)
\]

For the second option the solution (4.61) is valid for \( t \) in the range \( 0 < t < t_m \), while for \( t > t_m \):

\[
y(x, t) = kPF_2(x, t) \quad (4.62)
\]

where,

\[
F_2(x, t) = \frac{1}{(1 - r)} \left[ r e^{-\zeta_2} - e^{-\zeta_1} + z \left( e^{-\zeta_1} - e^{-\zeta_2} \right) - (z - 1) e^{-\zeta_1} + (z - r) e^{-\zeta_2} \right] \left[ -\frac{1}{2} \left( 1 - \frac{x^3}{L^3} \right) + \frac{3}{2} \left( 1 - \frac{x}{L} \right) \right] \quad (4.63)
\]
and $r$ and $z$ have been defined in Eq. (4.59). From Eqs. (4.60 - 4.63), it is observed that for a Kelvin-Voigt viscoelastic model, there are two unknown parameters $\tau_1$ and $\tau_2$. Now that the purely viscoelastic part of the solution has been obtained, it is noted that Eq. (4.60) vanishes at $t = 0$, as it should. There may already be a displacement in existence that is not equal to zero. To include this part of the solution, it is recalled that for displacements in linear elastic solids, which are governed by linear partial differential equations, that is, the Navier equations, a constant multiple of these equations added to a constant is also a solution. Thus a linear combination of Eq. (4.60) and (4.62) is

$$y(x, t) = kP \left[ \frac{1}{E} + \gamma_1 F_1(x, t) + \gamma_2 F_2(x, t) \right]$$

(4.64)

Because of the introduction of the parameters $\gamma_1$ and $\gamma_2$, the complete solution depends on five parameters: $\tau_1$, $\tau_2$, $z$, $\gamma_1$ and $\gamma_2$. If $\gamma_2 = 0$, then the solution of Eq. (4.64) is applicable for $P = \text{constant}$. Critchfield [29] used Zocher’s [30] application of the Correspondence Principle to generate curves for the time dependent tip deflection of an isotropic, homogeneous beam. These results are compared to the analytical results obtained from Eq. (4.64).

**Example 4.1**

A cantilevered isotropic beam as shown in Figure 4.4 is subjected to a tip load and has the following parameters. Obtain the time dependent tip deflection using Eq. (4.64) and compare to the solution obtained by Critchfield [29]:
P = load = 44.5 N  \quad L = beam length = 1.52 m
b = width of beam = 0.152 m  \quad 2c = height = 0.0254 m
E = Young’s Modulus = 206.8 GPa  \quad \nu = Poisson’s ratio = 0.3
\tau_1 = 15 \text{ sec}  \quad \tau_2 = 10.5 \text{ sec}  \quad r = 0.7

Using the above parameters, \( \eta_e \) and \( z \) are determined by using
\[
\eta_e = \frac{2\nu}{1-2\nu}, \quad z = \frac{1 + \eta_e}{1 + \eta_e} = \frac{(2 + 3\eta_e)r + 1}{3 + 3\eta_e}
\]

The viscoelastic parameters can be found by fitting the analytical data to the closed form solution obtained by Critchfield [29]. When \( \gamma_1 = 0.11 \times 10^{-3} \text{ Pa} \) and \( \gamma_2 = 0.12 \times 10^{-3} \text{ Pa} \) are chosen, the analytical curve generated from Eq. (4.64) compares very favorably to the experimental curve obtained from Ref. [29], as shown in Figure 4.5. In this figure, the load is applied for fifty seconds and then removed.

Figure 4.5: Time dependent tip deflection for isotropic beam using a K-V model.
4.3.2 Sandwich Beam

As shown in Figure 4.6, a cantilevered sandwich beam consisting of four layers is considered. The effective modulus is determined from Eq. (4.45) where \( N \) is four, and the core thickness is divided such that it consists of two layers, each having the same thickness as each face sheet. The remaining procedure is very similar to the methodology described in Section 4.3.1 for the isotropic beam.

As has been described by Gibson [4], for a symmetric laminate, the effective modulus \( E_f \) is given by Eq. (4.47). For a four ply layup, the effective modulus is given by

\[
E_f = \frac{1}{8}[E_1 + 7E_2]
\]

which in terms of Lame’ constants, is

\[
E_f = \frac{1}{8}\left[\frac{\mu_{e_1} (3\lambda_{e_1} + 2\mu_{e_1})}{\lambda_{e_1} + \mu_{e_1}} + 7 \frac{\mu_{e_2} (3\lambda_{e_2} + 2\mu_{e_2})}{\lambda_{e_2} + \mu_{e_2}}\right]
\]

This effective modulus is substituted into Eq. (4.56) and the Principle of Correspondence is applied. The time dependent tip deflection is determined to be
\[ y_{\text{tip}} = P k \left[ \frac{1}{E_t} + \frac{\theta_v \gamma_1}{d} \sum_{0}^{N} \frac{g(\alpha_k)}{p'(\alpha_k)} e^{\alpha_k t} \right] \]

where \( d \) is a constant and \( g(\alpha_k) \) and \( p(\alpha_k) \) are polynomials evaluated at the values, \( \alpha_k \) and \( p'(\alpha_k) \) is the derivative of \( p(\zeta) \) evaluated at \( \alpha_k \). The constants \( \alpha_k \) are the roots of \( p(s) = 0 \) with \( \alpha_1 = 0 \).

**Example 4.2**

For a sample calculation, (using properties obtained from Ref. [22] with subscript 1 representing the composite face sheet properties and 2 being the balsa core properties), the solution is shown in Figure 4.7:

\[
\begin{align*}
P &= 1112.1 \text{ N} & L &= 1.524 \text{ m} & c &= 0.01375 \text{ m} & b &= 0.1524 \text{ m} \\
E_1 &= 27.6 \text{ GPa} & E_2 &= 68.9 \text{ MPa} \\
\nu_1 &= 0.14, & \nu_2 &= 0.011 \\
\tau_{11} &= 5 \text{ sec} & \tau_{12} &= 2.5 \text{ sec} & \tau_{21} &= 1 \text{ sec} & \tau_{22} &= 0.5 \text{ sec} \\
\gamma_1 &= 0.11 \times 10^{-3} \text{ Pa}
\end{align*}
\]

![Composite Beam Tip Deflection K-V Model](image)

Figure 4.7: Sandwich cantilever beam tip deflection using a K-V model.
4.4 Discussion of Shear Effects

In the standard mechanics of materials approach, deformation due to shear stresses is often neglected due to their minimal contribution. A determining factor of shear effects is the beam’s thickness-to-span ratio, h/L. For isotropic beams, if this ratio is less than 1/10, the shear effects are negligible and are not considered. However, in composites, particularly sandwich beams, where the shear modulus of the core material is much less than the effective flexural modulus, transverse shear effects may play a key role. [4,31]. The solid beam is basically considered to be of an isotropic material due to its large number of ply groups as seen from its quasi-isotropic layup pattern ([0/90/45/-45]_{as}), and its h/L ratio of 1/60. Although the h/L ratio of the sandwich beam is 1/24, it was decided that it would be important to perform a check to determine the magnitude of the shear effects for this case as well as the homogeneous case.

4.4.1 Energy Method for Determination of Shear Effects

An energy method is used to determine the center deflection of a beam under the setup and loading shown in Figure 4.1. The strain energy due to both normal stresses and shear stresses are determined from the following equations which can be found in any Mechanics of Materials text such as Ref. [32]:

\[ U_\sigma = \int \frac{\sigma^2}{2E} \, dV \]  \hspace{1cm} (4.65a)

\[ U_\tau = \int \frac{\tau_{xy}^2}{2G} \, dV \]  \hspace{1cm} (4.65b)

where \( U_\sigma \) is the strain energy due to bending and \( U_\tau \) is the strain energy due to shear and
\( \sigma \) = normal stress, \( E \) = Young’s modulus
\( \tau_{xy} \) = shear stress, \( G \) = shear modulus
\( V \) = volume.

Once the total strain energy is determined, Castigliano’s theorem is used to determine the center deflection by using:

\[
y_c = \frac{\partial U_\sigma}{\partial R} + \frac{\partial U_\tau}{\partial R}
\]

where \( R \) is the load applied to the point where the deflection is required. For this study, since the center deflection is needed, \( R \) is a dummy load applied to the center of the beam as shown in Figure 4.8.

![Figure 4.8: Isotropic beam in four-point bending with dummy load R.](image)

4.4.1.1 Center deflection for homogeneous beam

The normal stress due to bending for a homogenous beam is expressed as

\[
\sigma = \frac{My}{I}
\]

where \( M \) is the bending moment, \( y \) is measured from the neutral axis of the cross-section and \( I \) is the moment of inertia. Substituting Eq. (4.67) into Eq. (4.65a) results in Eq.
(4.68), the familiar expression for the strain energy due to bending for a homogeneous beam.

\[
U_\sigma = \frac{1}{2} \int_0^L \frac{M^2}{2EI} \, dx \tag{4.68}
\]

Since the beam is symmetrically loaded, only the left half of the beam is used in the calculations. Using Castigliano’s theorem with Eq. (4.68), the center deflection for the beam due to bending is determined from

\[
y_c \bigg|_{ \text{bend} } = 2 \left[ \int_0^{L/3} \frac{M_1}{EI} \frac{\partial M_1}{\partial R} \, dx + \int_{L/3}^{L/2} \frac{M_2}{EI} \frac{\partial M_2}{\partial R} \, dx \right] \tag{4.69}
\]

where

\[
M_1 = \frac{1}{2} (P + R)x \quad 0 \leq x \leq \frac{L}{3}
\]

\[
M_2 = \frac{1}{2} \left( \frac{PL}{3} + Rx \right) \quad \frac{L}{3} \leq x \leq \frac{L}{2}
\]

After performing the integration in Eq. (4.69), R is set to 0, and the center deflection due to bending is determined to be

\[
y_c \bigg|_{ \text{bend} } = \frac{23}{1296} \frac{PL^3}{EI} \tag{4.71}
\]

The contribution from shear is now formulated. It is first noted that shear exists only in the outer segments of a beam loaded in four-point flexure. Due to symmetry, the segments contribute equally and the center deflection is expressed as

\[
y_c \bigg|_{ \text{shear} } = 2 \frac{\partial U_{s1}}{\partial R} \tag{4.72}
\]
The shear strain energy $U_{\gamma}$ is determined from Eq. (4.65b) and for a rectangular cross-section, the shear stress is expressed as

$$
\tau_{xy} = \frac{3 F_v}{2 A} \left(1 - \frac{y^2}{c^2}\right)
$$

(4.73)

where $F_v$ is the shear force, $A$ is the cross-sectional area and $2c$ is the total height as shown in Figure 4.9. Equation (4.73) is plotted and as seen from Figure 4.10, the distribution of shear stress in the top half of a transverse section of a rectangular beam is parabolic.

Figure 4.9: Solid beam cross-section.
Figure 4.10: Shear stress distribution in a solid rectangular cross-section.

To determine $\tau_{xy}$ from Eq. (4.73), the shear force must be determined, and for the loading shown in Figure 4.1, the shear force exists only in the outer segments and is

$$F_v = \frac{1}{2} (P + R) \quad 0 \leq x \leq \frac{L}{3}, \quad \frac{L}{3} \leq x \leq L$$

(4.74)

Setting $dV = b \ dy \ dx$ and substituting Eq. (4.73) into Eq. (4.65b), the strain energy due to shear is determined to be

$$U_{\text{ni}} = \frac{9}{8} G b \left[ \int_0^{L/3} \left( 1 - \frac{y^2}{c^2} \right) \ dy \right]$$

(4.75)

Substituting the results of Eq. (4.75) into Eq. (4.72), gives the center deflection due to shear as

$$y_c = \frac{PL}{5 \ G b h}$$

(4.76)

Taking into account the effect of both normal and shearing stresses, the center deflection is obtained by combining Eq. (4.76) and Eq. (4.71).
The first term in Eq. (4.77) is due to the bending stresses and the second term is the contribution due to the shear effects. As can be seen, both the modulus ratio $E/G$ and the thickness-to-span ratio play a role in determining the magnitude of the shear effects. As an example, the center deflection due to normal and shear stresses is calculated for the solid beam at a temperature of 93 °C and a load of 890 N. For the monolithic beam, an effective flexural modulus is first calculated from the experimental data and then Eq. (4.77) is used to determine the deflection due to both normal and shear stresses. The percent contribution from shear is found to be less than 0.3%.

4.4.1.2 Center deflection for sandwich beam

To determine the effect of normal and shear stresses for the sandwich beam, the strain energy must be computed using Eq. (4.65). Since the beam is a composite, consisting of composite face sheets and a core, the method of transformed sections is used as shown in Figure 4.11 [32].
This modified section is now considered to consist entirely of the face sheet material.

Using Figure 4.11, the transformed or effective moment of inertia is determined to be

\[
I_{\text{eff}} = \eta \frac{b}{12} (h - 2t)^3 + 2 \left[ \frac{1}{12} bt^3 + bt \left( \frac{h - t}{2} \right)^2 \right]
\]  

(4.78)

where

\[
\eta = \frac{E_C}{E_F}
\]  

(4.79)

and \(E_C\) is the longitudinal modulus of the core and \(E_F\) is the longitudinal modulus of the face sheets.

Since the strain energy from the two end sections is equal, integration need only be performed over the first section and then multiplied by a factor of two. Considering the middle section along with the end sections, the total strain energy due to the normal stress is
where

\[
\sigma_1 = \frac{M_1 y}{I_{\text{eff}}}
\]  \hspace{1cm} (4.80a)

\[
\sigma_2 = \frac{M_2 y}{I_{\text{eff}}}
\]  \hspace{1cm} (4.80b)

and \(M_1\) and \(M_2\) are defined in Eq. (4.70). The subscripts 1 and 2 signify the two domains, \([0 \leq x \leq L/3]\) and \([L/3 \leq x \leq L/2]\), respectively. The stresses in Eq. (4.80a) and Eq. (4.80b) describe the stress in the face sheets. Stresses in the core are obtained by multiplying these stresses by \(\eta\), the stiffness ratio given in Eq. (4.79). The normal stress in the center segment, \([L/3 \leq x \leq 2L/3]\), is plotted using Eq. (4.80b). Figure 4.12 shows the piecewise linear distribution of the normal stress in the core (broken line) and the face sheet (continuous line) in the top half of the cross-section. As expected, the bending moment is carried mainly by the face sheets.
The strain energy due to shear is determined from Eq. (4.65b). Since the transformed beam does not have a rectangular cross-section, nor is it homogeneous, the shear stress is expressed as

$$\tau = \frac{F_v Q}{It}$$

where $F_v$ is the shear force defined in Eq. (4.74). For the sandwich beam, the first moment $Q$ must be determined for the core and the face sheets and it is defined as

$$Q = \int_{y_1}^{c} y \, dA$$

where $Q$ represents the first moment with respect to the neutral axis of the shaded areas shown in Figure 4.13.
Figure 4.13: Transformed sections to determine Q for (1) core and (2) face sheets.

The first moment for the core and the face sheets is determined to be

\[
Q_1 = \frac{1}{2} b \left\{ (h-t)t + \left( \frac{h}{2} - t - y \right) \left( \frac{h}{2} - t + y \right) \eta \right\}
\]

\[
Q_2 = \frac{1}{2} b \left( \frac{h}{2} - y \right) \left( \frac{h}{2} + y \right)
\]

The shear stresses in the upper half of the transformed cross-section are

\[
\tau_1 = \frac{F_y Q_1}{I_{eff}(\eta b)} \quad 0 \leq y \leq \left( \frac{h}{2} - t \right)
\]

\[
(4.81)
\]

\[
\tau_2 = \frac{F_y Q_2}{I_{eff}(b)} \quad \left( \frac{h}{2} - t \right) \leq y \leq \frac{h}{2}
\]

Equation (4.81) describes the shear stress in the face sheets. To obtain the shear stress in the core, the stress in Eq. (4.81) is multiplied by the moduli ratio, \( \eta \). Using Eq. (4.81), the shear stress distribution is plotted for \( 0 \leq y \leq h/2 \). From Figure 4.14, it is seen that at
y = h/2, the stress is zero, as expected. Also, at the interface, the stress is the same for the core and face sheets. This stress distribution is quite different from the distribution obtained for a rectangular homogeneous cross-section shown in Figure 4.10.

Figure 4.14: Shear stress distribution in sandwich beam.

The shear strain energy can now be determined by using Eq. (4.65b). Since the shear force in the middle section of the beam is zero, and equal for the first and last segments, integration need only be in the range 0 ≤ x ≤ L/3 and the result multiplied by a factor of two. Due to symmetry, the shear stress need only be evaluated in the top half of the cross-section and then multiplied by two to obtain the strain energy for the entire cross-section. The shear strain energy is determined to be
\[
U_{\tau} = 4b \left\{ \frac{h-1}{2} \int_0^h \left( \frac{1\left(\eta \tau_1\right)^2}{G_1} \right) \, dy + \frac{h-1}{2} \int_0^h \left( \frac{1\left(\tau_2^2\right)}{G_2} \right) \, dy \right\} \int_0^1 \, dx \right\},
\]

(4.82)

The center deflection is obtained from Castigliano’s theorem (Eq. 4.66) and determined to be

\[
y_{\sigma} = \frac{\partial U_{\sigma}}{\partial R} = \frac{13PL^3}{54be_c e_f \left(3h^2 - 6ht + 4t^2\right)} \left(2E_c \left(t^2 \left(h - 2t\right) + E_c \left(h - 2t\right)^2 \eta^2\right)\right)
\]

\[
y_{\tau} = \frac{\partial U_{\tau}}{\partial R} = \frac{PL^3}{5be_c e_f \left(2t \left(3h^2 - 6ht + 4t^2\right) \left(h - 2t\right)^3 \eta^2\right)} \left(2G_c t^3 \left(10h^2 - 15ht + 6t^2\right) \left(h - 2t\right) \left(30 \left(h - t\right)^2 t^2 \left(h - t\right) \eta + \left(h - t\right)^4 \eta^2\right)\right)
\]

where \(y_{\sigma}\) and \(y_{\tau}\) are the center deflections due to normal stresses and shear stresses, respectively. Using the dimensions for the sandwich beams used in this study, and using the material properties for the sandwich beam at 121 °C, the contribution due to shear is calculated to be approximately 6%. Although the contributions due to shear for the solid beam are negligible, these effects should be taken into consideration for the sandwich beam. For this study, deformation due to shear is not directly included in the analysis, but shear effects are included in the results, due to using initial conditions from the experimental data. Shear deformation should be included in the analytical solution in the future. Although not directly implemented in the flexure deflection equation obtained in this study, the next section presents a shear deformation method that can be incorporated in the analysis for future work.
4.4.2 Energy Method for Anisotropic Laminates for Inclusion of Shear Effects

For inclusion of shear deformation in the analysis, another shear analysis that is investigated is a procedure performed by Whitney [31] for anisotropic laminates. The complete development of this method for a four-point bend test is given in Reference [31]. Using this method for the beam shown in Figure 4.1, the resulting deflection equation incorporating shear effects is determined to be

\[
y(x) = \frac{P L^3}{1296 E_f I} \left[ -27 \left( \frac{x}{L} \right)^2 + 27 \left( \frac{x}{L} \right) - 1 + \frac{54}{k} \frac{E_f}{G_{xz}} \frac{I}{L^4 b h} \right]
\]  
(4.83)

where \( E_f \) is the effective flexural modulus and \( G_{xz} \) is the shear modulus. The factor \( k \) is a function of the ply properties and stacking sequence of composite laminates [33]. If a monolithic beam with a rectangular cross-section is considered, a shear correction factor \( S \) is introduced as

\[
S = \frac{1}{k} \left( \frac{E_f}{G_{xz}} \right) \left( \frac{h}{L} \right)^2
\]  
(4.84)

and Eq. (4.83) is now expressed as

\[
y(x) = \frac{P L^3}{1296 E_f I} \left[ -27 \left( \frac{x}{L} \right)^2 + 27 \left( \frac{x}{L} \right) - 1 + \frac{27}{6} S \right]
\]  
(4.85)

As before, from Eq. (4.84) it can be seen that the magnitude of the shear factor \( S \) is dependent on the thickness-to-length ratio as well as the modulus ratio. The deflection at the center can be determined by setting \( x = L/2 \) in Eq. (4.85) and is given by
Eq. (4.86) is used in Section 4.4.3 to demonstrate how shear effects can be implemented in the analysis.

4.4.3 Analysis with Shear Effects

If the shear effects must be considered, the same procedure of applying the Principle of Correspondence is used. The center deflection is expressed as

\[ y_c = \frac{PL^3}{1296E_fI} \left[ 23 + \frac{27}{6} S \right] \] (4.86)

\[ \gamma_2 = \frac{E_f}{G_{xy}} \left[ 23 + \frac{27}{6} \right] \] (4.87)

where

\[ \gamma_1 = \frac{1}{1296} \frac{L^3}{I} \]

and the effective flexural modulus and the shear modulus are expressed as previously defined in Eq. (4.46).

\[ E_f = \theta_0 \left[ \theta_1 E_{c1} + \theta_2 E_{c2} \right] \]

\[ G_{xy} = \varphi_0 \left[ \varphi_1 G_{c1} + \varphi_2 G_{c2} \right] \]

Writing in terms of Lame’ parameters,
\[ E_f = \theta_o \left[ \theta_1 \frac{\mu_{el} (3\lambda_{el} + 2\mu_{el})}{\lambda_{el} + \mu_{el}} + \theta_2 \frac{\mu_{e2} (3\lambda_{e2} + 2\mu_{e2})}{\lambda_{e2} + \mu_{e2}} \right] \] (4.88)

\[ G = \varphi \mu_{el} + \varphi_2 \mu_{e2} \]

The principle of correspondence is now applied to Eq. (4.86) for the viscoelastic case and \( E_f \) is replaced by \( \psi_1(\zeta) \) and \( G \) is replaced by \( \psi_2(\zeta) \) resulting in

\[ \bar{y}_c = \frac{y_1P}{\psi_1(\zeta)} \left[ 23 + 18\gamma_2 \frac{\psi_1(\zeta)}{\psi_2(\zeta)} \right] \] (4.89)

where

\[ \psi_1(\zeta) = \theta_o \left[ \theta_1 \frac{\mu_1(\zeta) (3\lambda_1(\zeta) + 2\mu_1(\zeta))}{\lambda_1(\zeta) + \mu_1(\zeta)} + \theta_2 \frac{\mu_2(\zeta) (3\lambda_2(\zeta) + 2\mu_2(\zeta))}{\lambda_2(\zeta) + \mu_2(\zeta)} \right] \] (4.90)

\[ \psi_2(\zeta) = \varphi \left[ \varphi_1 \mu_1(\zeta) + \varphi_2 \mu_2(\zeta) \right] \]

Using Eq. (4.51) and (4.52), \( \psi_1(\zeta) \) and \( \psi_2(\zeta) \) can now be expressed as

\[ \psi_1(\zeta) = \theta_o \left[ \theta_1 E_{el} H_1(\zeta) + \theta_2 E_{e2} H_2(\zeta) \right] \]

\[ \psi_2(\zeta) = \varphi \left[ \varphi_1 \mu_1 H_1(\zeta) + \varphi_2 \mu_2 H_2(\zeta) \right] \] (4.91)

Therefore,

\[ \frac{\psi_1(\zeta)}{\psi_2(\zeta)} = \frac{\theta_o \left[ \theta_1 E_{el} H_1(\zeta) + \theta_2 E_{e2} H_2(\zeta) \right]}{\varphi \left[ \varphi_1 \mu_1 H_1(\zeta) + \varphi_2 \mu_2 H_2(\zeta) \right]} \] (4.92)

The ratio given in Eq. (4.92) is for an anisotropic beam in the Laplace domain, but the Laplace inversion procedure for this is quite involved. However, if a homogeneous, isotropic case is considered, then an approximation can be made such that \( H_1(\zeta) \approx H_2(\zeta) \).

Then, the ratio simply reduces to a constant, namely the ratio of the moduli. This is a further validation of the Principle of Correspondence.
\[
\frac{\psi_1(\zeta)}{\psi_2(\zeta)} = \frac{E_f}{G}
\]

**4.5 Determination of constants \(n_o\) and \(n\)**

The procedure for obtaining the viscoelastic constants is shown in the flow chart in Figure B.8, Appendix B. It must first be noted that the dimension of \(n_o\) is [1/sec]. Therefore, its inverse \(1/n_o\) represents a time scale. In the context of a viscoelastic material, the parameter \(1/n_o\) is the relaxation time factor. Referring to Eq. (4.59), if the material follows a K-V model, the following ratios are the obvious relaxation times (the remaining ratios in Eq. (4.59) are dimensionless).

\[
\frac{\mu_{vc}}{\mu_e}, \quad \frac{\lambda_{vc}}{\lambda_e}, \quad \frac{3\lambda_{vc} + 2\mu_{vc}}{3\lambda_e + 2\mu_e}
\]

where \(\lambda_{vc}\) and \(\mu_{vc}\) are constants and are explicit in the model itself. On the other hand, constants \(n\) and \(n_o\), both of dimension [1/sec], are implicit in the definition of the spectrum function \(\varphi(\alpha)\).

To obtain \(n_o\) and then \(n\), available experimental creep data under a constant stress is used. Eq. (4.14) is restated as

\[
e_{11}(t) = \frac{\sigma}{E_e} + \frac{n}{\pi} \int_0^t e_{11}(t-\tau) \psi(\tau)d\tau \tag{4.93}
\]

where

\[
\psi(t) = \int_{n_o t}^{\infty} \frac{\cos(z - n_o t)}{z} dz \tag{4.94}
\]

The following algorithm is used in this study to determine the values of \(n\) and \(n_o\).
1. As a first approximation of the Volterra equation (Eq. (4.93), the following form is taken

\[ e^{(1)}_{11}(t) = \alpha + \beta e^{-n_{o}t} \]  

(4.95)

where

\[ e_{11}(t_{f}) - \frac{\sigma}{E_{e}e^{-n_{o}t_{f}}} \]

\[ \alpha = \frac{\beta}{1 - e^{-n_{o}t_{f}}}, \quad \beta = e_{11}(0) - \alpha \]

where \( t_{f} \) is the final time at which \( e_{11} \) is known from experimental data. Also, from Eq. (4.93), it should be noted that

\[ e_{11}(0) = \frac{\sigma}{E_{e}} \]

At another time \( t = t_{m} \), the strain value \( e_{11}(t_{m}) \) is also known. The following transcendental equation is formed and solved numerically for \( n_{o} \).

\[ \alpha + \beta e^{-n_{o}t_{m}} = e_{11}(t_{m}) \]  

(4.96)

2. Having determined \( n_{o} \), the function \( \psi(t) \) in Eq. (4.94) is generated.

3. Once \( \psi(t) \) and \( n_{o} \) are known, Eq. (4.95) is used at \( t = t_{f} \) to obtain \( n \) by

\[ n = \frac{\pi}{t_{f}} \left[ e_{11}(t_{f}) - \frac{\sigma}{E_{e}} \right] \]

\[ \frac{1}{t_{f}} \int_{0}^{t_{f}} e_{11}(t_{f} - \tau) \psi(\tau) d\tau \]  

(4.97)

4. The second approximation can now be obtained for any number of \( t \) values by using the equation

\[ e^{(2)}_{11}(t) = \frac{\sigma}{E_{e}} + \frac{n}{\pi} \int_{0}^{1} e^{(1)}_{11}(t - \tau) \psi(\tau) d\tau \]  

(4.98)
where $e_{i1}^{(1)}(t)$ is the first approximation and $e_{i1}^{(2)}(t)$ is the second approximation that is being calculated. This new value of $e_{i1}^{(2)}(t)$ is then used in Eq. (4.96) and the second approximation of $n_0$ and then $n$ is obtained. Based on the numerical solution, the differences of successive values of $n_0$ and $n$ are very small.
CHAPTER V
RESULTS and DISCUSSION

The data used in this study is obtained from two separate experimental programs. The first program was implemented at the University of Wyoming [21], and it supplied all the tensile and tensile creep data that has been used in this study. Data for the four-point flexure tests is obtained from a study undertaken at the Naval Surface Warfare Center, Carderock Division [22]. All analytical computations are accomplished using MathCad.

Table 5.1 summarizes the test matrix and the fabrication methods from the experimental programs that are used for this research.

Table 5.1: Fabrication Characteristics of Test Pieces

<table>
<thead>
<tr>
<th>Test</th>
<th>Specimen Type</th>
<th>Material Fiber</th>
<th>Resin System</th>
<th>Layup Sequence</th>
<th>Fabrication Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile [21]</td>
<td>Resin</td>
<td>-</td>
<td>Derakane 510</td>
<td>-</td>
<td>Casting</td>
</tr>
<tr>
<td>Tensile [21]</td>
<td>Laminate</td>
<td>E-glass Plain weave</td>
<td>Derakane 510</td>
<td>Woven roving: [0-90]</td>
<td>VARTM</td>
</tr>
<tr>
<td>Flexure 4-point [22]</td>
<td>Laminate</td>
<td>E-glass 24 oz/yd²</td>
<td>Derakane 510A</td>
<td>[0/90/45/-45]₄s balanced weave</td>
<td>VARTM</td>
</tr>
<tr>
<td>Flexure 4-point [22]</td>
<td>Sandwich</td>
<td>E-glass face sheets/ Balsa core [h = 2 in, 9 lb/ft³]</td>
<td>Derakane 510A</td>
<td>[0/90/45/-45/ core]₅ balanced weave</td>
<td>VARTM</td>
</tr>
</tbody>
</table>
5.1 Tensile Testing Results and Discussion

The temperature dependent properties of the tensile resin and composite specimens are listed in Table 5.2 and Table 5.3, respectively. The tables also show the test data that is used for this study in terms of the temperature and the stress level. The properties shown in this table are the elastic properties that are obtained from the quasi static tension-to-failure tests. The data used in this study is obtained from isothermal tests where the temperature is held constant [21].

As described in the algorithm in Section 4.5, the value for \( n_o \) is determined from solving the transcendental equation given in Eq. (4.96). Tables 5.2 and 5.3 also list the values of the viscoelastic parameters, \( n \) and \( m \), that are obtained for the various resin and composite tensile creep test cases.

Table 5.2: Temperature Dependent Resin Properties

<table>
<thead>
<tr>
<th>Temp (°C)</th>
<th>Stress (MPa)</th>
<th>E (GPa)</th>
<th>( v )</th>
<th>( n \times 10^5 ) (1/sec)</th>
<th>( m = \frac{n}{n_o} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>5</td>
<td>3.820</td>
<td>0.373</td>
<td>6.20</td>
<td>0.651</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td>0.215</td>
</tr>
<tr>
<td></td>
<td>33</td>
<td></td>
<td></td>
<td></td>
<td>0.391</td>
</tr>
<tr>
<td>66</td>
<td>5.5</td>
<td>2.440</td>
<td>0.312</td>
<td>29.58</td>
<td>1.797</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td></td>
<td></td>
<td>30.86</td>
<td>1.848</td>
</tr>
<tr>
<td>93</td>
<td>0.8</td>
<td>1.497</td>
<td>0.342</td>
<td>46.35</td>
<td>1.743</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td></td>
<td>43.02</td>
<td>1.779</td>
</tr>
<tr>
<td>121</td>
<td>0.1</td>
<td>0.016</td>
<td>0.467</td>
<td>6.91</td>
<td>0.201</td>
</tr>
</tbody>
</table>
Table 5.3: Temperature Dependent Composite Properties

<table>
<thead>
<tr>
<th>Temp (°C)</th>
<th>Stress (MPa)</th>
<th>E (GPa)</th>
<th>ν</th>
<th>n x 10^5 (1/sec)</th>
<th>m = \frac{n}{n_o}</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>172 255</td>
<td>28.51</td>
<td>0.135</td>
<td>2.52 1.87</td>
<td>0.085 0.064</td>
</tr>
<tr>
<td>66</td>
<td>172 255 296</td>
<td>27.34</td>
<td>0.140</td>
<td>0.94 3.21 4.19</td>
<td>0.111 0.110 0.145</td>
</tr>
<tr>
<td>93</td>
<td>131</td>
<td>26.80</td>
<td>0.100</td>
<td>2.35</td>
<td>0.157</td>
</tr>
<tr>
<td>121</td>
<td>90 131</td>
<td>17.31</td>
<td>0.280</td>
<td>13.90 17.90</td>
<td>0.633 0.463</td>
</tr>
<tr>
<td>149</td>
<td>90 131</td>
<td>11.5</td>
<td>0.32</td>
<td>10.11 88.95</td>
<td>0.481 0.257</td>
</tr>
</tbody>
</table>

5.1.1 Time Dependent Strain

The time dependent strain is calculated by first determining \(n_o\) by choosing the initial time, final time, and an intermediate time. The intermediate time which produced the value of \(n_o\) that best fit the experimental data is selected. This value of \(n_o\) is then used to calculate \(n\) and the remaining properties.

Figure 5.1 shows the variation of strain with time for resin specimens at 24 °C. These figures are generated solely on the value of \(n_o\) and the boundary values of the strain for \(t = 0\) sec and \(t = t_{final}\) seconds.
As can be seen from this figure, the match between the experimental and the analytically generated data is very good. At 24 °C, viscoelastic effects are not pronounced, especially at the lower stress level of 5 MPa.

Figures 5.2 and 5.3 depict the variation of strain in resin samples at 66 °C and 93 °C, respectively. From these figures, it is seen that the methodology successfully duplicates the experimental data for each test case.
Figure 5.2: Resin creep response in longitudinal tension - 66 °C.

Figure 5.3: Resin creep response in longitudinal tension - 93°C.

As expected, the composite creep response shown in Figure 5.4 at 24 °C shows even less viscoelastic effects when compared to its resin counterpart. But, the viscoelastic behavior
is seen to increase at elevated temperatures as shown in Figure 5.5. A complete set of creep responses of all test pieces are shown in Figures C.1- C.18, Appendix C.

Figure 5.4: Composite creep response in longitudinal tension - 24 °C

Figure 5.5: Composite creep response in longitudinal tension - 121 °C
5.1.2 Time Dependent Compliance

Recalling Eq. (4.31) from Chapter 4, it is observed that the creep compliance is simply the strain divided by the stress. The stress is taken to be constant for creep experiments and therefore the appropriate constant stress value for each specimen is used to generate the analytical curves. For the experimental testing, the strain for each time is divided by the stress at that time. For that reason, the stress curves and the compliance curves corresponding to that stress level are shown together. Much of the difference between the analytical and experimental curves is due to the varying stress level. Figure 5.6 shows the stress curve and the corresponding compliance for resin at 66 °C at a stress level of 5 MPa. It is noted that the stress level is not maintained at a constant value of 5 MPa. The corresponding experimental compliance follows the trend of the varying stress level, whereas the analytical solution used a constant value of 5 MPa to obtain the compliance.

Figure 5.6: Resin stress and compliance response - 66 °C, 5 MPa.
Figure 5.7 depicts the stress and compliance for the resin at 66 °C, but at a stress level of 14 MPa. The stress level is maintained at 14 MPa, and the experimental and analytical curves match quite well.

The stress and compliance plots for all the resin samples are shown in Appendix C, Figures C.19 – C.34, and all show the same trend.

Figure 5.8 shows the stress and corresponding compliance of a composite sample at 66 °C at a stress level of 255 MPa.
Figure 5.8: Composite stress and compliance response - 66 °C, 255 MPa

The stress level is maintained at 255 MPa and the analytically generated compliance curve matches the experimental data quite well. This is true for most of the composite specimens as shown in Figures C.35 – C.54, Appendix C.

5.1.3 Time Dependent Modulus

Having obtained the values for n and m for each test case, the modulus as a function of time is generated by Eq. (4.19). This equation is evaluated directly, without making any approximations. The elastic properties given in Table 5.2 for the resin and Table 5.3 for the composite are used in Eq. (4.19) to generate the modulus curves. It is well known that the elastic stiffness of polymer-matrix composite materials degrades at elevated temperatures [1]. As stated in Section 4.1.5, the modulus cannot be obtained by simply inverting the compliance and, therefore, is not directly obtainable via experiments. Using this methodology, a modulus response can be obtained. Figures 5.9 and 5.10 demonstrate the modulus response of a resin and a composite sample, respectively.
Figure 5.9: Resin modulus response - 66 °C, 14 MPa.

Figure 5.10: Composite modulus response - 66 °C, 172 MPa.
From these figures, it can be seen that the loss in modulus of the resin is far greater than
that of the composite. This is expected since the polymer component is the viscoelastic
component of the material system. Figures C.55 – C.72 in Appendix C show the
modulus response of all the resin and composite specimens.

Table 5.4 lists the percent difference in the minimum and maximum values of the
analytically obtained modulus and Poisson’s ratio values.

Table 5.4: Percent change in analytical Modulus and Poisson’s ratio

<table>
<thead>
<tr>
<th>Temp (°C)</th>
<th>Stress (MPa)</th>
<th>% Modulus decrease</th>
<th>%Poisson’s ratio increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resin</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>5</td>
<td>22.30</td>
<td>7.02</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>10.26</td>
<td>3.37</td>
</tr>
<tr>
<td></td>
<td>33</td>
<td>15.15</td>
<td>4.89</td>
</tr>
<tr>
<td>66</td>
<td>5</td>
<td>69.95</td>
<td>29.22</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>71.96</td>
<td>29.80</td>
</tr>
<tr>
<td>93</td>
<td>0.8</td>
<td>70.85</td>
<td>24.32</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>71.79</td>
<td>24.56</td>
</tr>
<tr>
<td>121</td>
<td>0.1</td>
<td>8.40</td>
<td>0.59</td>
</tr>
<tr>
<td>Composite</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>172</td>
<td>3.45</td>
<td>8.51</td>
</tr>
<tr>
<td></td>
<td>255</td>
<td>2.58</td>
<td>6.52</td>
</tr>
<tr>
<td>66</td>
<td>172</td>
<td>3.65</td>
<td>8.56</td>
</tr>
<tr>
<td></td>
<td>255</td>
<td>4.47</td>
<td>9.59</td>
</tr>
<tr>
<td></td>
<td>296</td>
<td>5.86</td>
<td>13.06</td>
</tr>
<tr>
<td>93</td>
<td>131</td>
<td>5.96</td>
<td>19.17</td>
</tr>
<tr>
<td>121</td>
<td>90</td>
<td>25.15</td>
<td>16.39</td>
</tr>
<tr>
<td></td>
<td>131</td>
<td>20.01</td>
<td>13.52</td>
</tr>
<tr>
<td>149</td>
<td>90</td>
<td>18.97</td>
<td>9.59</td>
</tr>
<tr>
<td></td>
<td>131</td>
<td>12.89</td>
<td>6.74</td>
</tr>
</tbody>
</table>
The analytical modeling correctly depicts the larger decrease in the resin modulus when compared to the decrease of the composite specimen modulus. Minor viscoelastic effects are noted for the resin at 121 °C due to the application of the small (0.1 MPa) load. This is also shown in Figure 5.11.

![Resin Strain vs. Time](Figure 5.11: Resin strain response - 121 °C, 0.1 MPa)

Observing the data for the composite in Table 5.4, it is seen that the degradation in the modulus increases significantly at 121 °C (there is a modulus drop of 25.15%). A larger degradation in the moduli is expected for the composite at 149 °C, but from viewing the strain response in Figure 5.12 it is seen that the data is usable and duplicated only to about 3500 seconds for the 90 MPa loading and about 300 seconds for the 131 MPa loading. For the sake of completeness this data is analyzed although no additional information is gained.
Figure 5.12: Composite strain response - 149 °C, 90 MPa and 131 MPa.

5.1.4 Time Dependent Poisson’s Ratio

Since only the longitudinal strain was recorded for the resin specimens, no experimental Poisson’s ratio was available. For the composite specimens, Poisson’s ratio was obtained by using extensometers to monitor and record both the longitudinal and transverse strains. Strain gages were not used because these must be bonded to the test article. For elevated temperature environment, the adhesive used for bonding was cured at a temperature above the test temperature. To avoid subjecting the test piece to any elevated temperature before testing, extensometers were employed. Much of the transverse gage data proved to be unusable due to testing issues regarding the transverse strain. Especially at higher temperatures (at least 93 °C), the specimens become “soft”, and there are gripping issues to consider. For these composite specimens, much of the transverse strain data (producing Poisson’s ratios greater than 0.5, even greater than 1) is discarded. Also, perhaps due to transverse matrix/ fiber failures producing jumps, some of the data could not be used even at the lower temperatures. However, using Eq. (4.24), the analytical Poisson’s ratio is generated for all test articles. The solution uses the elastic...
Poisson’s ratio and the elastic Young’s Modulus that is listed in Tables 5.2 and 5.3.

Figure 5.13 shows the Poisson’s ratio response for resin at 66 °C, 5 MPa. The remaining figures produced similar plots and are shown in Figures C.73 – C.90. Table 5.4 gives an indication of the trend of the Poisson’s ratio for various temperatures and stress levels.

Figure 5.13: Resin Poisson’s ratio - 66 °C, 5 MPa

Equation 4.24 produces an increasing type function. However, Lakes [26] states that the Poisson’s ratio can increase or decrease and some specimens can have an increasing/decreasing type cycle.

The experimental data in Figure 5.14 is questioned since minor viscoelastic effects are observed for the composite samples at room temperature. The Poisson’s ratio obtained from tensile tests at room temperature established the parameter at 0.14. This figure shows a Poisson’s ratio of 0.08 upon application of the load and the value stays fairly constant.
Figure 5.14: Composite Poisson’s ratio - 24 °C, 172 MPa

In Figure 5.15, the comparison between the experimental and analytical data is shown for a composite sample at 66 °C at a stress level of 296 MPa. The trend is similar and the comparison is favorable. Again, had the initial value of the Poisson’s ratio from the creep data been used as the Poisson’s ratio at $t = 0$ instead of the elastic value obtained from the tensile tests, the curve would be an excellent match. A complete set of the Poisson’s ratio response of the composite samples can be found in Appendix C, Figures C.73- C.90.
5.2 Flexure Test Results and Discussion

Table 5.5 summarizes the loading matrix for the flexure data. With respect to Figure 4.1, the load levels specified in Table 5.5 represent the total mechanical load, or P. The temperature dependent properties of the components are given in Tables 5.6 and 5.7. As stated before, the following experimental data and properties are obtained from Ref. [22] and the balsa properties are assumed to be constant with respect to temperature. Properties obtained from Ref. [22] do not correlate very well with the experimental data and Ref. [22] states that necessary data was not available at the time of the research and therefore, many properties were extrapolated. For this study, wherever possible, the experimental data is used to obtain the necessary properties, i.e., effective flexure moduli. Otherwise, properties obtained from Ref. [22] are used in the computations.
Table 5.5: Flexure Test Matrix [22]

<table>
<thead>
<tr>
<th>Temp (°C)</th>
<th>Load (N)</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>93</td>
<td>890</td>
<td>Monolithic</td>
</tr>
<tr>
<td>121</td>
<td>890</td>
<td>Monolithic</td>
</tr>
<tr>
<td>149</td>
<td>445</td>
<td>Monolithic</td>
</tr>
<tr>
<td>93</td>
<td>3114</td>
<td>Sandwich</td>
</tr>
<tr>
<td>121</td>
<td>3114</td>
<td>Sandwich</td>
</tr>
</tbody>
</table>

Table 5.6: Flexure Specimens, Composite Elastic Properties [22]

<table>
<thead>
<tr>
<th>Temp (°C)</th>
<th>$E_{1}$, $E_{2}$ (GPa)</th>
<th>$G_{12}$ (GPa)</th>
<th>$\nu_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>27.6</td>
<td>4.1</td>
<td>0.14</td>
</tr>
<tr>
<td>93</td>
<td>22.1</td>
<td>3.3</td>
<td>0.14</td>
</tr>
<tr>
<td>121</td>
<td>15.2</td>
<td>2.3</td>
<td>0.14</td>
</tr>
<tr>
<td>149</td>
<td>8.3</td>
<td>1.2</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Table 5.7: Flexure Specimens, Balsa Elastic Properties [22]

<table>
<thead>
<tr>
<th>$E_{1}$, $E_{2}$ (MPa)</th>
<th>$E_{3}$ (GPa)</th>
<th>$G_{12}$ (MPa)</th>
<th>$G_{13}$ (GPa)</th>
<th>$\nu_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>68.9</td>
<td>2.8</td>
<td>13.8</td>
<td>0.15</td>
<td>0.011</td>
</tr>
</tbody>
</table>

5.2.1 Monolithic Beam

The center deflection for a four-point flexure test is given by Eq. (4.44). The same algorithm as given in Section 4.5 and used for the tensile specimens in Section 5.1
is used to determine the viscoelastic constants \( n \) and \( m \) for the flexure tests. The obtained values are listed in Table 5.8.

**Table 5.8: Monolithic Beam Viscoelastic Constants.**

<table>
<thead>
<tr>
<th>Temp (°C)</th>
<th>Load (N)</th>
<th>( y_c(t_f) )</th>
<th>( n \times 10^4 ) (1/sec)</th>
<th>( m = \frac{n}{n_o} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>93</td>
<td>890</td>
<td>15.2</td>
<td>10.3</td>
<td>2.266</td>
</tr>
<tr>
<td>121</td>
<td>890</td>
<td>21.3</td>
<td>8.04</td>
<td>2.485</td>
</tr>
<tr>
<td>149</td>
<td>445</td>
<td>11.4</td>
<td>11.4</td>
<td>2.187</td>
</tr>
</tbody>
</table>

The comparison of the center deflection from experimental data and analytical data is shown in Figures 5.16 – 5.18. In Table 5.8, \( y_c(t_f) \) is the center deflection at the final time, \( t_f \) which is approximately 14,000 seconds. To generate the curves, the initial deflection is taken to be zero at time \( t = 0 \) and the final deflection point at \( t_f \) is used. As seen from the figures, the analytical curves compare quite favorably with the experimental data.
Figure 5.17: Monolithic beam center deflection response - 121 °C, 890 N.

Figure 5.18: Monolithic beam center deflection response - 149 °C, 445 N.
5.2.2 Composite Beam

For the composite beam, either consisting of an anisotropic layup and/ or consisting of more than one material, the procedure discussed in Section 4.2.2 is followed. To determine the time dependent deflection for a composite beam, an effective flexural modulus must first be determined. Once this is expressed, the Correspondence Principle is applied and the deflection as a function of time is determined following the same procedure as was followed for the isotropic solid beam. The following section elaborates on how the coefficients in Eq. (4.46) are determined.

5.2.2.1 Effective Flexural Modulus

A half section of the cross-section of the composite sandwich beam is shown in Figure 5.19. This beam is treated as a layered beam consisting of a total of ten plies. The composite face sheet is treated as a single component of thickness equal to 0.635 cm. A total of ten layers are obtained for this cross-section when the core thickness is expressed in equivalent number of plies relative to the composite face sheets.

![Figure 5.19: Half section of composite sandwich beam.](image)
Applying Equation 4.45 for N = 10, Eq. (5.1) is obtained.

\[ E_f = \frac{8}{10^5} [64E_c + 61E_L] \]  

(5.1)

In Eq. 5.1, \( E_c \) is the modulus of the core and \( E_L \) is the longitudinal Young’s Modulus of the laminate or the face sheets. Upon comparison of Eq. (5.1) and Eq. (4.46), the following is obtained:

\[ \kappa_0 = 0.125 \]
\[ \kappa_1 = 64 \]
\[ \kappa_2 = 61 \]

5.2.2.2 Composite Beam Deflection

To obtain the time dependent deflection, the Volterra equation in Eq. (4.55) is evaluated. Using the same procedure that is followed for the solid beam, the viscoelastic constants for the sandwich beam are determined. The difference in the two cases is that for the sandwich beam, instead of determining \( n_{o1} \) and \( n_{o2} \) as is required for Eq. (4.53), a simpler approach is to obtain effective constants, \( n_{\text{eff}} \) and \( m_{\text{eff}} \) from the evaluation of Eq. (4.55). Table 5.9 lists the effective constants obtained for the sandwich beam cases. In Table 5.9, \( y_c(t_f) \) is the center deflection at the final time, \( t_f \) and Figures 5.20 and 5.21 show the plot of the experimental and the analytically obtained center deflection for the sandwich beam.

Both Figures 5.20 and 5.21 are generated using the boundary conditions that at \( t = 0 \) sec, the deflection is zero and at the final time \( t_f \) the final deflection (listed in Table 5.9) is used. It is seen from these figures that the experimental and analytical curves are in very good agreement.
Table 5.9: Sandwich Beam Viscoelastic Constants.

<table>
<thead>
<tr>
<th>Temp (°C)</th>
<th>Load (N)</th>
<th>(y_c(t_f)) (mm)</th>
<th>(n_{eff} \times 10^4) (1/sec)</th>
<th>(m_{eff})</th>
</tr>
</thead>
<tbody>
<tr>
<td>93</td>
<td>3114</td>
<td>8.4</td>
<td>9.34</td>
<td>2.197</td>
</tr>
<tr>
<td>121</td>
<td>3114</td>
<td>12.2</td>
<td>8.69</td>
<td>2.255</td>
</tr>
</tbody>
</table>

Figure 5.20: Sandwich beam center deflection response - 93 °C, 3114 N.

Figure 5.21: Sandwich beam center deflection response - 121 °C, 3114 N.
CHAPTER VI
DISCUSSION AND CONCLUSIONS

6.1 Discussion of Results

The spectrum function $\varphi(\alpha)$ is successfully used to develop the fundamental Lame’ functions. These functions are then used to express all mechanical properties of interest. Using the Principle of Correspondence, creep deflections, strains, modulus, compliance and Poisson’s ratio are determined in the Laplace domain. The inversion process results in the corresponding numerical solutions in the time domain which are compared to the available experimental data. For time dependent strain and deflection, the resulting Volterra equation is solved. A transcendental equation is then solved from which the viscoelastic constant $n_o$ is determined. The obtained constants $m$ and $n$ are used to determine the remaining properties for each temperature and stress level. To assess the effectiveness of the model, the generated data is compared to the experimental data.

It is not concluded that the chosen spectrum function $\varphi(\alpha)$ is unique. In fact, other functions may also model the data quite well. This particular choice of $\varphi(\alpha)$ produces a clean, closed-form solution for the time dependent material properties. The obtained constants, $m$ and $n$ are unique and if another function for $\varphi(\alpha)$ is chosen, then the
The viscoelastic constants \( n \) and \( m \) seem to describe the trend in the viscoelastic behavior. The degradation of modulus in the composite specimens is much less than that of the resin pieces and this is reflected by the magnitude of the obtained constants. It is seen that for the resin test pieces, \( m \) generally seems to be greater than 1, whereas for the composite pieces, it seen to be less than 1. The findings seem to indicate that smaller \( m \) values indicate a smaller degree of viscoelastic behavior. The constants do vary with temperature and stress level, but their range of variation is small, and they can be used to predict viscoelastic properties for a particular material system.

From Table 5.4, it is seen that the variation in Poisson’s ratio for resin specimens is much larger than that of the composite specimens. For the composite specimens, Poisson’s ratio seems to be a weak function to time. Majority of the experimental curves of the time dependent Poisson’s ratio would have compared very favorably with the analytical curves had the initial Poisson’s ratio of the creep experiments been used instead of the elastic Poisson’s ratio obtained from the quasi-static tensile testing. Both
the experimental data and the curves obtained from the approximation show a minimal change in Poisson’s ratio for all temperatures and stress levels for both resin and composite test pieces.

For all the flexure tests, the analytically generated data matches the experimental data very well. From Tables 5.8 and 5.9, it is seen that there is not much variation between the constants for various temperatures or stress levels for the solid beams and the sandwich beams in flexure. Also there is not much difference between the constants obtained for the solid beams and the effective constants of the sandwich beams. Deducing from the results of tensile testing, the constants obtained in flexure may simply reflect the degree of viscoelastic behavior. Although shear deformation was not directly included in the analytical modeling, the effects of shear deformation do appear in the results due to using the initial and final deflection data.

This procedure provides a means by which the modulus as well as the Poisson’s ratio can be obtained. The constants seem to reflect the level of viscoelastic behavior. The constants m and n obtained from the algorithm seem to correctly map trends as well as magnitudes.

6.2 Conclusions

Linear viscoelastic behavior is modeled through a new approach involving the development of the viscoelastic Lamé’ functions. Traditionally, modeling of viscoelastic behavior has been done by combining simple elements, i.e. Kelvin-Voigt (K-V) or Maxwell, to express the behavior in terms of exponential terms. However, not all materials obey the characteristics of these simple elements. In this study, a new spectrum
function is used that not only is able to describe general viscoelastic behavior, but it is also able to model the simple behavior of Kelvin-Voigt solids. The methodology is developed for homogenous, isotropic materials and extended to composite materials by determining their effective time dependent properties.

Data obtained from the experimental tensile testing of glass fiber reinforced polymer laminates and their polymer constituent is used for comparison with the analytically obtained time dependent strain and compliance. Also, using this methodology, properties that are either not computable from experimental testing (modulus) or difficult to measure (Poisson’s ratio) are modeled.

The analytical solution for the time dependent deflection is also obtained for beams in four-point flexure and this solution is compared to the experimentally obtained data. From this modeling, viscoelastic constants are computed and used to determine the remaining properties. For demonstration purposes, the study determined the deflection curve for a cantilevered beam using a K-V model. Comparisons between the analytical data and the experimental data are very favorable.

For determination of shear effects, an energy method is used and the level of shear deformation is determined for a monolithic beam as well as a sandwich beam. Although not used directly in the analytical solution, an energy method is used to demonstrate how shear effects can be included in the analysis. In this study, by using the initial conditions, the role of shear deformation is incorporated.

The theory is based on sound mathematical principles of linear elasticity and viscoelasticity and can easily be extended to the non-isothermal case. The power of the Correspondence Principle is exploited, resulting in the time dependent properties of
interest, producing a model that correlates very well with the available experimental data. Since the viscoelastic parameters are purely functions of time, this theory can also be used for two and three dimensional problems.

To further validate the relationship between the obtained viscoelastic constants and the material systems, additional experiments are necessary. More precise data is needed, especially for the Poisson’s ratio at elevated temperatures. Various composite layups need to be tested in tension to establish the use of the effective properties in composites. Shear deformation, especially for sandwich beams, needs to be included in future work. More accurate materials properties need to be obtained for flexure analysis.
REFERENCES


APPENDIX A

MATHEMATICAL FORMULATIONS
A. Dirac Delta Function and its properties:

1. \[ \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \]

2. \[ \delta(t) = \frac{dH(t)}{dt}, \quad H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \]

3. \[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

4. \[ \int_{-\infty}^{\infty} f(t) \delta(t) \, dt = f(0) \quad \text{or} \quad \int_{0}^{\infty} f(t) \delta(t) \, dt = \begin{cases} f(0) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases} \]

5. \[ \int_{-\infty}^{\infty} f(t) \delta(t - \tau) \, dt = f(t) \]

B. Definitions / Identities

1. \[ \text{si}(n_o t) = -\frac{\pi}{2} + \text{Si}(n_o t) \]

2. \[ \text{ci}(n_o t) = \int_{n_o t}^{\infty} \frac{\cos z}{z} \, dz \]

3. \[ \text{Ci}(n_o t) = -\int_{n_o t}^{\infty} \frac{\cos z}{z} \, dz \quad \Rightarrow \quad \text{ci} = -\text{Ci} \]

4. \[ \text{Si}(n_o t) = \int_{0}^{n_o t} \frac{\sin z}{z} \, dz \]

5. \[ \text{si}(n_o t) = -\int_{n_o t}^{\infty} \frac{\sin z}{z} \, dz \]

6. \[ \sin(a \pm b) = \sin a \cos b \pm \sin b \cos a \]

7. \[ \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \]
C. Useful Laplace Transforms

1. \( \mathcal{L} [F(t)] = \overline{F(\zeta)} = \int_{0}^{\infty} F(t) e^{-\zeta t} dt \)  
   \quad \text{Definition of Laplace Transform}

2. \( \mathcal{L}^{-1} [\overline{F(\zeta)}] = F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\zeta) e^{\zeta t} d\zeta \)  
   \quad \text{Definition of Inverse Laplace}

3. \( \overline{F(\zeta) G(\zeta)} = \mathcal{L} \left[ \int_{0}^{t} F(\tau) G(t-\tau) d\tau \right] \)  
   \quad \text{Laplace Convolution}

4. \( \mathcal{L} [\delta(t)] = \int_{0}^{t} [\delta(t) e^{-\zeta t}] dt = 1 \)

5. \( \mathcal{L} [\sin \alpha t] = \frac{\alpha}{\zeta^2 + \alpha^2} \)

6. \( \mathcal{L} [\cos \alpha t] = \frac{\zeta}{\zeta^2 + \alpha^2} \)

7. \( \mathcal{L} [\text{Si}(t) \cos(t) - \text{Ci}(t) \sin(t)] = \frac{\ln \zeta}{\zeta^2 + 1} \)

8. \( \mathcal{L} [f(rt)] = \mathcal{L} \left[ \int_{0}^{\infty} e^{-\zeta t} f(rt) dt \right] = \frac{1}{r} F \left( \frac{\zeta}{r} \right) \)

9. \( \mathcal{L} [\text{si}(t) \cos(t) + \text{ci}(t) \sin(t)] = -\frac{\pi}{2} \frac{\zeta}{\zeta^2 + 1} + \frac{\ln \zeta}{\zeta^2 + 1} \)
Figure B.1: General theory flow chart.

\[ e_{11} = \frac{1}{E_c} T_{11} = \frac{\lambda_e + \mu_e}{\mu_e (3\lambda_e + 2\mu_e)} T_{11} \]

Apply the Correspondence Principle

\[ \bar{e}_{11} = \frac{\lambda(\zeta) + \mu(\zeta)}{\mu(\zeta)[3\lambda(\zeta) + 2\mu(\zeta)]} \bar{T}_{11} \]

Input the functions

\[ \lambda(\zeta) = \lambda_e + \zeta \bar{\lambda}(\zeta) \]
\[ \mu(\zeta) = \mu_e + \zeta \bar{\mu}(\zeta) \]

Refer to \( \lambda_e(t) \) and \( \mu_e(t) \) flow chart

Inverse Laplace transform.

Objective:
Determine \( \lambda_e(t) \) and \( \mu_e(t) \)

Refer to \( \lambda_e(t) \) and \( \mu_e(t) \) flow chart
1. Eringen[20]

\[ \lambda_v(t) = -\lambda_c \int_0^{\infty} \phi(\alpha) \left( 1 - e^{-\alpha t} \right) d\alpha \]

2. Chosen spectrum function

\[ \phi(\alpha) = \frac{n}{\pi (1 + n_o^2 \alpha^2)} \]

3. Input \( \phi(\alpha) \)

\[ \lambda_v(t) = -\lambda_c \int_0^{\infty} \frac{n}{\pi (1 + n_o^2 \alpha^2)} \left( 1 - e^{-\alpha t} \right) d\alpha \]

4. Integrate to obtain \( \lambda_v(t) \)

\[ \lambda_v(t) = -\frac{\lambda_c}{2} m \left[ 1 - \frac{2}{\pi} \int_{n_o}^{\infty} \frac{\sin(z - n_o t)}{z} dz \right] \]

\[ m = \frac{n}{n_o} \]

\[ \int_{n_o}^{\infty} \frac{\sin(z - n_o t)}{z} dz = c_i(n_o t) \sin(n_o t) + \sin(n_o t) \cos(n_o t) \]

5. Take Laplace Transform of \( \lambda_v(t) \).

\[ \mathcal{L} \left[ \lambda_v(t) \right] = \lambda_v(\zeta) = -\lambda_c \left[ \frac{m}{2\zeta} + \frac{m}{\pi n_o} \left( -\frac{\pi}{2} \frac{\zeta/n_o}{\left( \zeta/n_o \right)^2 + 1} + \ln \frac{\zeta/n_o}{\left( \zeta/n_o \right)^2 + 1} \right) \right] \]

6. Form the function

\[ \lambda(\zeta) = \lambda_c + \zeta \lambda_v(\zeta) \]

\[ \lambda(\zeta) = \lambda_c \left[ 1 - \frac{m}{2} \right] - \tilde{f} \left( \frac{\zeta}{n_o} \right) \]

\[ \tilde{f} \left( \frac{\zeta}{n_o} \right) = \frac{m}{\pi} \left[ \frac{\zeta/n_o \ln \left( \frac{\zeta/n_o}{n_o} \right)}{\left( \frac{\zeta/n_o}{n_o} \right)^2 + 1} - \frac{\pi}{2} \frac{\left( \frac{\zeta/n_o}{n_o} \right)^2}{\left( \frac{\zeta/n_o}{n_o} \right)^2 + 1} \right] \]

Figure B.2: Viscoelastic functions flow chart.
\[ e_{11} = \frac{1}{E_e} T_{11} = \frac{\lambda_e + \mu_e}{\mu_e (3\lambda_e + 2\mu_e)} T_{11} \]

Apply the Correspondence Principle

\[ \bar{e}_x(\zeta) = \frac{\lambda(\zeta) + \mu(\zeta)}{\mu(\zeta)[3\lambda(\zeta) + 2\mu(\zeta)]} \bar{T}_x \]

Refer to \( \lambda_\tau(t) \) flow chart

Input \( \lambda(\zeta) \) and \( \mu(\zeta) \)

\[ \bar{e}_x(\zeta) = \frac{1}{E_e} \left[ \frac{1 - m}{2} - \bar{f} \left( \frac{\zeta}{n_o} \right) \right] \bar{T}_x \]

Obtain the Laplace Inverse

\[ \frac{n}{\pi} \int_0^t e_x(t - \tau) \psi(\tau) \, d\tau = e_x(t) - \frac{\sigma}{E_e} \]

Volterra Integral Equation

Figure B.3: Strain, e(t) flow chart.
**Elastic Young’s modulus**

\[ E_e = \frac{\mu_e (3\lambda_e + 2\mu_e)}{\lambda_e + \mu_e} \]

**Memory function**

\[ \overline{E}(\zeta) = \frac{9\overline{G}(\zeta)\overline{K}(\zeta)}{3\overline{K}(\zeta) + \overline{G}(\zeta)} \]

**Correspondence Principle**

\[ \frac{\mu(\zeta)[3\lambda(\zeta) + 2\mu(\zeta)]}{\lambda(\zeta) + \mu(\zeta)} \]

**Input the Laplace transform of \( K(t) \) and \( G(t) \)**

\[ E_e \left[ \left(1 - \frac{m}{2} \right) \cdot \overline{f} \left( \frac{\zeta}{n_o} \right) \right] \]

\[ \overline{E}(\zeta) = \frac{E_e}{\zeta} \left[ \left(1 - \frac{m}{2} \right) \cdot \overline{f} \left( \frac{\zeta}{n_o} \right) \right] \]

Obtain the inverse Laplace.

\[ E(t) = E_e \left[ \left(1 - \frac{m}{2} \right) + \frac{m}{\pi} \int_{n_o}^{\infty} \frac{\sin (z - n_o t)}{z} dz \right] \]

Figure B.4: Modulus, \( E(t) \) flow chart.
1. Elastic bulk modulus

\[ K_e = \frac{E_e}{3(1 - 2\nu_e)} \]

2. Solve for the elastic Poisson’s ratio

\[ \nu_e = \frac{1}{2} - \frac{1}{6} \frac{E_e}{K_e} \]

3. Apply the Correspondence Principle

\[ \zeta \nu(\zeta) = \frac{1}{2} - \frac{1}{6} \frac{E(\zeta)}{K(\zeta)} \]

4. Inverse Laplace

\[ \nu(t) = \frac{1}{2} - \frac{1}{6} \int_0^t E(t - \tau) \frac{dB(\tau)}{d\tau} d\tau \]

5. Lake’s approximation [26]

\[ \nu(t) = \frac{1}{2} - \frac{1}{6} \frac{E(t)}{K_e} \]

6. Input \( E(t) \) and obtain the time dependent Poisson’s ratio.

\[ \nu(t) = \frac{1}{2} - \frac{3\mu_e}{6(\lambda_e + \mu_e)} \left[ \left( 1 - \frac{m}{2} \right) + \frac{m}{\pi} \int_{n_o t}^{\infty} \frac{\sin(z - n_o t)}{z} dz \right] \]

Figure B.5: Poisson’s ratio, \( \nu(t) \) flow chart.
1. Creep bulk modulus

\[ K(t) = \lambda_e + \frac{2}{3} \mu_e + \lambda_\nu(t) + \frac{2}{3} \mu_\nu(t) \]

2. Apply Correspondence Principle.

\[ \overline{K}(\zeta) = \frac{1}{\zeta} \left[ \lambda(\zeta) + \frac{2}{3} \mu(\zeta) \right] \]

3. Creep bulk compliance in the Laplace domain.

\[ \overline{B}(\zeta) = \frac{1}{\zeta^2 \overline{K}(\zeta)} \]

4. Substitute (2) into (3).

\[ B(\zeta) = \frac{3}{\zeta (3 \lambda_e + 2 \mu_e) \left[ 1 - \frac{m_1}{2} \right] \tilde{f} \left( \frac{\zeta}{n_0} \right)} \]

5. Obtain the Laplace inverse.

\[ B(t) = \frac{mn_o}{\pi} \int_0^t B(t - \tau) \psi(\tau) \, d\tau = \frac{3}{3 \lambda_e + 2 \mu_e} \]

*Volterra integral equation.*

Figure B.6: Bulk compliance, B( t) flow chart.
1. Elastic deflection equation.

\[ y_c = \frac{\beta}{E_e} P \]

\[ \beta = \frac{23}{1296} \frac{L^3}{I} \]

2. Apply the Correspondence Principle.

\[ \bar{y}_c(\zeta) = \beta \left[ \frac{\lambda(\zeta) + \mu(\zeta)}{\mu(\zeta)3\lambda(\zeta) + 2\mu(\zeta)} \right] \bar{P}(\zeta) \]

3. Input \( \lambda(\zeta) \) and \( \mu(\zeta) \).

\[ \bar{y}_c(\zeta) = \beta \frac{1}{E_e} \frac{1}{\left(1 - \frac{m}{2}\right) - \tilde{f} \left(\frac{\zeta}{n_o}\right)} \bar{P}(\zeta) \]

4. Obtain the Laplace inverse.

\[ y_c(t) = \frac{n}{\pi} \int_0^t y_c(t - \tau) \psi(\tau) d\tau = \frac{\beta}{E_e} P(t) \]

*Volterra integral equation.*

Figure B.7: Center deflection, \( y_c(t) \) flow chart.
1. Volterra equation for strain.

\[ e_{11}(t) = \frac{\sigma}{E_e} + \frac{n}{\pi} \int_0^t e_{11}(t-\tau) \psi(\tau) \, d\tau \]

3. Form the 1\textsuperscript{st} approximation and use the initial conditions.

\[ \beta = e_{11}(0) - \alpha \]

\[ e_{11}^{(1)}(t) = \alpha + \beta e^{-n \cdot t} \]

\[ e_{11}(t_f) = \frac{\sigma}{E_e} e^{-n \cdot t_f} \]

\[ \alpha = \frac{1}{1 - e^{-n \cdot t_f}} \]

3. Form the transcendental equation for an intermediate time and solve for \( n_0 \).

\[ \alpha + \beta e^{-n \cdot t_m} = e_{11}(t_m) \]

4. Determine \( \psi(t) \)

\[ \psi(t) = \int_{n \cdot t}^{\infty} \frac{\cos(z - n_0 t)}{z} \, dz \]

5. Determine \( n \)

\[ n = \frac{\pi}{\int_{t_f}^{t_f} e_{11}(t_f) - \frac{\sigma}{E_e} \right) \]

\[ n = \frac{\pi}{\int_0^{t_f} e_{11}(t_f - \tau) \psi(\tau) \, d\tau} \]

6. Use the obtained constants \( n_0 \) and \( n \) to determine other properties or form the second approximation.

\[ e_{11}^{(2)}(t) = \frac{\sigma}{E_e} + \frac{n}{\pi} \int_0^t e_{11}^{(1)}(t-\tau) \psi(\tau) \, d\tau \]

Figure B.8: Determination of viscoelastic constants
APPENDIX C

TENSILE CREEP PLOTS
Figure C.1:  Resin creep response in longitudinal tension - 24 °C, 5 MPa

Figure C.2:  Resin creep response in longitudinal tension - 24 °C, 13 MPa
Figure C.3: Resin creep response in longitudinal tension - 24 °C, 33 MPa

Figure C.4: Resin creep response in longitudinal tension - 66 °C, 5 MPa
Figure C.5: Resin creep response in longitudinal tension - 66 °C, 14 MPa

Figure C.6: Resin creep response in longitudinal tension - 93 °C, 0.8 MPa
Figure C.7: Resin creep response in longitudinal tension - 93 °C, 3 MPa

Figure C.8: Resin creep response in longitudinal tension - 121 °C, 0.1 MPa
Figure C.9: Composite creep response - 24 °C, 172 MPa

Figure C.10: Composite creep response - 24 °C, 255 MPa
Figure C.11: Composite creep response - 66 °C, 172 MPa

Figure C.12: Composite creep response - 66 °C, 255 MPa
Figure C.13: Composite creep response - 66 °C, 296 MPa

Figure C.14: Composite creep response - 93 °C, 131 MPa
Figure C.15: Composite creep response - 121 °C, 90 MPa

Figure C.16: Composite creep response - 121 °C, 131 MPa
Figure C.17: Composite creep response - 149 °C, 90 MPa

Figure C.18: Composite creep response - 149 °C, 131 MPa
Figure C.19: Resin stress response in longitudinal tension - 24 °C, 5 MPa

Figure C.20: Resin compliance response in longitudinal tension - 24 °C, 5 MPa
Figure C.21: Resin stress response in longitudinal tension - 24 °C, 13 MPa

Figure C.22: Resin compliance response - 24 °C, 13 MPa
Figure C.23: Resin stress response in longitudinal tension - 24 °C, 33 MPa

Figure C.24: Resin compliance response - 24 °C, 33 MPa
Figure C.25: Resin stress response in longitudinal tension - 66 °C, 5 MPa

Figure C.26: Resin compliance response - 66 °C, 5 MPa
Figure C.27: Resin stress response in longitudinal tension - 66 °C, 14 MPa

Figure C.28: Resin compliance response - 66 °C, 14 MPa
Figure C.29: Resin stress response in longitudinal tension - 93 °C, 0.8 MPa

Figure C.30: Resin compliance response - 93 °C, 0.8 MPa
Figure C.31: Resin stress response in longitudinal tension - 93 °C, 3 MPa

Figure C.32: Resin compliance response - 93 °C, 3 MPa
Figure C.33: Resin stress response in longitudinal tension - 121 °C, 0.1 MPa

Figure C.34: Resin compliance response - 121 °C, 0.1 MPa
Figure C.35: Composite stress response - 24 °C, 172 MPa

Figure C.36: Composite compliance response - 24 °C, 172 MPa
Figure C.37: Composite stress response - 24 °C, 255 MPa

Figure C.38: Composite compliance response - 24 °C, 255 MPa
Figure C.39: Composite stress response - 66 °C, 172 MPa

Figure C.40: Composite compliance response - 66 °C, 172 MPa
Figure C.41: Composite stress response - 66 °C, 255 MPa

Figure C.42: Composite compliance response - 66 °C, 255 MPa
Figure C.43: Composite stress response - 121 °C, 296 MPa

Figure C.44: Composite compliance response - 66 °C, 296 MPa
Figure C.45: Composite stress response - 93 °C, 131 MPa

Figure C.46: Composite compliance response - 93 °C, 131 MPa
Figure C.47: Composite stress response - 121 °C, 90 MPa

Figure C.48: Composite compliance response - 121 °C, 90 MPa
Figure C.49: Composite stress response - 121 °C, 131 MPa

Figure C.50: Composite compliance response - 121 °C, 131 MPa
Figure C.51: Composite stress response - 149 °C, 90 MPa

Figure C.52: Composite compliance response - 149 °C, 131 MPa
Figure C.53: Composite stress response - 149 °C, 131 MPa

Figure C.54: Composite compliance response - 149 °C, 131 MPa
Figure C.55: Resin modulus response in longitudinal tension - 24 °C, 5 MPa

Figure C.56: Resin modulus response in longitudinal tension - 24 °C, 13 MPa
Figure C.57: Resin modulus response in longitudinal tension - 24 °C, 33 MPa

Figure C.58: Resin modulus response in longitudinal tension - 66 °C, 5 MPa
Figure C.59: Resin modulus response in longitudinal tension - 66 °C, 14 MPa

Figure C.60: Resin modulus response in longitudinal tension - 93 °C, 0.8 MPa
Figure C.61: Resin modulus response in longitudinal tension - 93 °C, 3 MPa

Figure C.62: Resin modulus response in longitudinal tension - 121 °C, 0.1 MPa
Figure C.63: Composite modulus response - 24 °C, 172 MPa

Figure C.64: Composite modulus response - 24 °C, 255 MPa
Figure C.65: Composite modulus response - 66 °C, 172 MPa

Figure C.66: Composite modulus response - 66 °C, 255 MPa
Figure C.67: Composite modulus response - 66 °C, 296 MPa

Figure C.68: Composite modulus response - 93 °C, 131 MPa
Figure C.69: Composite modulus response - 121 °C, 90 MPa

Figure C.70: Composite modulus response - 121 °C, 131 MPa
Figure C.71: Composite modulus response - 149 °C, 90 MPa

Figure C.72: Composite modulus response - 149 °C, 131 MPa
Figure C.73: Resin Poisson’s ratio - 24 °C, 5 MPa

Figure C.74: Resin Poisson’s ratio - 24 °C, 13 MPa
Figure C.75: Resin Poisson’s ratio - 24 °C, 33 MPa

Figure C.76: Resin Poisson’s ratio - 66 °C, 5 MPa
Figure C.77: Resin Poisson’s ratio - 66 °C, 14 MPa

Figure C.78: Resin Poisson’s ratio - 93 °C, 0.8 MPa
Figure C.79: Resin Poisson’s ratio - 93 °C, 3 MPa

Figure C.80: Resin Poisson’s ratio - 121 °C, 0.1 MPa
Figure C.81: Composite Poisson’s ratio - 24 °C, 172 MPa

Figure C.82: Composite Poisson’s ratio - 24 °C, 255 MPa
Figure C.83: Composite Poisson’s ratio - 24 °C, 5 MPa

Figure C.84: Composite Poisson’s ratio - 66 °C, 255 MPa
Figure C.85: Composite Poisson’s ratio - 66 °C, 296 MPa

Figure C.86: Composite Poisson’s ratio - 93 °C, 131 MPa
Figure C.87: Composite Poisson’s ratio - 121 °C, 90 MPa

Figure C.88: Composite Poisson’s ratio - 121 °C, 131 MPa
Figure C.89: Composite Poisson’s ratio - 149 °C, 90 MPa

Figure C.90: Composite Poisson’s ratio - 149 °C, 131 MPa