CRITICAL PROPERTIES OF SMALL WORLD ISING MODELS

By

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In this dissertation, the critical scaling behavior of magnetic Ising models with long range interactions is studied. These long range interactions, when imposed in addition to interactions on a regular lattice, lead to small-world graphs. By using large-scale Monte Carlo simulations, together with finite-size scaling, the critical behavior of a number of different models is obtained. The Ising models studied in this dissertation include the $z$-model introduced by Scalettar, standard small-world bonds superimposed on a square lattice, and physical small-world bonds superimposed on a square lattice. From the scaling results of the Binder 4th order cumulant, the order parameter, and the susceptibility, the long-range interaction is found to drive the systems behavior from Ising-like to mean field, and drive the critical point to a higher temperature. It is concluded that with a large amount of strong long-range connections (compared to the interactions on regular lattices), so the long-range connection density is non-vanishing, systems have mean field behavior. With
a weak interaction that vanishes for an infinite system size or for vanishing density of long-range connections the systems have Ising-like critical behavior. The crossover from Ising-like to mean-field behavior due to weak long-range interactions for systems with a large amount of long-range connections is also discussed. These results provide further evidence to support the existence of physical (quasi-) small-world nanomaterials.
DEDICATION

To my parents.
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### List of Symbols, Abbreviations, and Nomenclature

**$A(T, M)$** Helmhotz potential

**$A_{CH}, A_{C_H}$** the critical amplitudes associate with the isomagnetic field specific heat

**$A_H$** critical isothermal amplitude

**$A_m$** prefactor in the scaling of the order parameter $|M|$ for weak small-world interaction system

**$A_t$** prefactor for $\tilde{T}_c$ and $T_c$ relation

**$A_\beta$** critical amplitudes associate with the order parameter $|M|

**$A_{\gamma}, A_{\gamma}$** critical amplitudes associate with the isothermal susceptibility $\chi_T$

**$A_{\xi}, A_{\xi}$** critical amplitudes associate with the correlation length

**$A_\chi$** prefactor in the scaling of the susceptibility $\chi$ for weak small-world interaction system

**$a$** lattice cell spacing

**$a_0$** best-fit prefactor for $f'_u$

**$a_H$** scaling parameter for magnetic field

**$a_t$** scaling parameter for reduced temperature

**$a(z)$** Scalettar’s scaling parameter

**$b(z)$** Scalettar’s scaling parameter

**$C$** specific heat density

**$C(t, L)$** zero-field specific heat density of an infinite size system

**$C_{H}(t, 0)$** zero-field isomagnetic field specific heat density

**$C_H$** isomagnetic field specific heat

**$C_M$** isomagnetization specific heat

**$d$** system dimension
$E$ free energy
$E_k$ internal energy per spin for the $k$th trial
$F(S, H)$ enthalpy
$f_C$ scaling function for the specific heat
$f_M$ scaling function for the order parameter $|M|$
$f_M$ scaling function for the magnetization
$f_Q$ scaling function for the Binder order cumulant $Q$
$f_U$ scaling function for the Binder $4$th order cumulant $U_4$
$f_u$ scaling function for the Binder $4$th order cumulant $U_4$
$f'_u$ derivative of $f_u$
$f_{\chi}$ scaling function for the susceptibility
$G(T, H)$ Gibbs potential
$G(t, 0)$ zero-field Gibbs potential density
$G(t, h)$ Gibbs potential density
$g$ crossover scaling variable
$g$ Gibbs potential density
$H$ external magnetic field
$\mathcal{H}$ Hamiltonian
$H_c$ magnetic field at the critical point
$H_{\text{eff}}$ effective magnetic field in mean field theory
$H_m$ magnetic field seen by the other spins in addition to the external magnetic field $H$
$h$ scaled external magnetic field
$h$ reduced magnetic field, $h = \tanh(\beta H)$
$i, j$ site index
$\langle ij \rangle$ spin pairs
interaction between spins on the nearest neighbor sites

interaction via small-world bonds

interaction between nearest neighbor small-world spins

interaction strength between spin $i$ and $j$

Boltzmann’s constant

linear size of a lattice system

Landau expansion coefficient

average separation between spins or sites

expansion coefficient

magnetization, or magnetization density

order parameter, the absolute value of the magnetization density

zero-field magnetization, spontaneous magnetization

zero-field magnetization density

magnetization density

zero-field magnetization density of an infinite size system

value of $M$ at zero temperature and zero field

isothermal magnetization

magnetization per spin for the $k$th trial

magnetization density

magnetic moment per spin at the critical point

spin magnetization, $m_i = \pm 1$ in two level magnetic systems

total spins in a lattice system

total spins in a system

number of trials

general parameter
p  ration of the SW bonds to regular bonds
p  strength ratio of the long-range interaction to the short-range interaction
Q  Binder order cumulant
R  effective interaction range
r  distance
r_{ij}  small-world bond length between spins i and j rounded to the upper integer
S  thermodynamic entropy
\langle S \rangle  statistic average of quantity S, \sum_{k} S_k e^{-\beta/k_B T}, S_k is the value of S at state k; in simulation, \langle S \rangle = \frac{1}{N_{\text{trial}}} \sum_{k} S_k, where S_k is the value of S at the kth trial
s  magnetic spin, s = \pm 1 in two level magnetic systems
s_i, s_j  spins at site i and j
s_{ik}  spin values at site i for the kth trial
T  temperature
\tilde{T}  reduced temperature \tilde{T} \equiv \frac{T}{T_c}
T_c  critical temperature
T_{c}  critical temperature of a pure square lattice Ising system
\tilde{T}_c  critical temperature of a weak small-world interaction system
t  reduced temperature t \equiv \frac{T - T_c}{T_c} = \frac{T}{T_c} - 1
U(S, M)  internal energy
U_4  Binder 4th order cumulant
u_\infty  U_4 value at critical point for infinite size system
V  lattice system volume
w  number of long-range (small-world) connections (shortcuts)
y_h  scaling exponent for h
y_t  scaling exponent for t
the number of average nearest neighbors

$Z, Z(T, H)$ partition function

$\alpha$ zero-field critical exponent associated with the isomagnetic field specific heat

$\alpha_H \left( \frac{\partial M}{\partial T} \right)_H$

$\alpha_M \left( \frac{\partial H}{\partial T} \right)_M = -\frac{\alpha H}{\chi_T}$

$\beta$ inverse temperature $\beta = 1/k_B T$

$\beta$ zero-field critical exponent associated with order parameter

$\Gamma(t, h)$ pair correlation function

$\Gamma(r, T), \Gamma(r, t)$ pair correlation function

$\gamma$ zero-field critical exponent associated with isothermal susceptibility $\chi_T$

$\delta$ critical isothermal exponent

$\eta$ pair correlation exponent

$\lambda$ a general parameter

$\lambda$ a parameter used to define the critical exponent of a thermal function

$\lambda$ ratio of spin caused magnetic field to the magnetization in mean field theory

$\nu$ zero-field critical exponent associated with the correlation length $\xi$

$\xi$ correlation length

$\xi(t, L)$ correlation length defined in a finite size lattice

$\xi_\infty$ correlation length of an infinite size system

$\sigma$ reduced magnetization

$\chi$ susceptibility density

$\chi(t, L)$ zero-field susceptibility density of an infinite size system

$\chi_T$ isothermal susceptibility

$\chi_T(t, 0)$ zero-field isothermal susceptibility density

$\chi_S$ adiabatic susceptibility
\( \phi \) ratio of small-world bonds to original nearest neighbor bonds

\( \Phi(T, H, L) \) thermodynamic potential

\( \phi(t, h, L) \) thermodynamic potential density
CHAPTER I

INTRODUCTION

Theoretical analysis and simulations are used to study critical phenomena. Some aspects of critical behavior in Ising-like spin systems are still not completely understood and still attract attention. One important concern is the crossover between the Ising-like behavior, which is characterized by short-range interactions, and mean-field critical behavior, which is characterized by long-range interactions. If the interaction range of the particles in a system varies, then a crossover occurs. It is very difficult to theoretically predict the quantitatively accurate crossover scaling functions and critical-point exponents. Hence Monte Carlo simulations are powerful tools to study critical behavior of a system. Finite Size Scaling (FSS) [58] is an important concept in magnetic Monte Carlo studies to obtain the critical behavior from the Monte Carlo data.

Small world networks were introduced to statistical physics right after the famous work of Watts and Strogatz [65] on networks. Magnetic systems with small world connections may have different physical properties, especially in their critical behavior and may introduce novel physics. Thus what such systems are, and what properties and dynamics they have are significant questions that need to be better understood. Many works in this area were presented, but these questions are still far from completely understood.
In this work, the critical behavior of small-world magnetic systems is studied. Three different small-world Ising models are constructed, and static Monte Carlo simulations are used to study the static properties of the three models. Finite size scaling results show systems with a large amount of long-range connections have mean field behavior, and Ising-like behavior is important for systems with a small amount of long-range connections or small-world connections built from spin chains. This dissertation is organized as follows. Chapter II is a brief review of critical phenomena, the Ising model, the mean field theory, Landau’s classical theory, finite size scaling, small world networks, Monte Carlo simulations, parallel programming, and uncorrelated random number generation. In chapter III, a \( z \) nearest connection Ising model, the \( z \)-model, is described and its critical behavior is analyzed. In chapter IV, results for a two-dimensional small-world model are presented and analyzed. Small world systems with strong long-range interaction and a large amount of long-range connections, so the long-range connection density is non-vanishing, are discussed and compared to the \( z \)-model; systems with a small amount of long-range connections, so the long-range connection density is vanishing, are discussed and compared to the regular two-dimensional Ising lattice. Also small world systems with non-vanishing density weak long-range interactions are presented and analyzed as well as its crossover properties from Ising-like critical behavior to mean-field behavior. Later, a modified two-dimensional physical small-world, in which the small-world bonds are physical spin-chains, is studied. The conclusions about the critical behavior of small-world magnetic systems are made in Chapter V.
CHAPTER II

REVIEW

This chapter contains a review of the background of critical phenomena, phase transitions, scaling functions for Ising models [23], mean field theory [63], Landau’s classic theory [63], finite-size scaling [57] [58], and small world networks [65] [50] [49] [64] [51] [47] [48], as well as parallelized Monte Carlo simulation.

2.1 Phase transitions and critical phenomena in magnetic systems

Critical phenomena [57] [63] were first studied scientifically more than a hundred years ago. In 1869, Andrews [2] discovered that the properties of the liquid and of the vapor become indistinguishable and the system shows a critical opalescence at a peculiar point when he was studying the properties of a carbon dioxide system. Several decades later, Pierre Curie [11] discovered the critical transition of a ferromagnetic system of iron and related it to the phenomena discovered by Andrews. In 1937, Landau [32] [33] proposed a general form explaining these phenomena theoretically. His model, which is called the mean-field model, qualitatively describes reasonable approximations to the transitions in both fluids and magnets. Since the 1960’s, much work has been done on scaling relations among critical exponents. Widom [68] [67] and Domb and Hunter [13] proposed
mathematical hypothesis in 1965 and Kadanoff [26] introduced heuristic arguments for the scaling problem. After the renormalization-group theory was invented and applied to thermal critical phenomena by Wilson [69], it became possible to easily explain the critical behavior of thermodynamic systems. This is particularly true for systems related to the Ising model [15], which includes uniaxial ferromagnets, simple fluids, and polymers.

In the last few decades, critical phenomena have attracted a lot of attention, and many new concepts have been introduced and various methods have been developed in order to understand the critical behavior near the critical point. For the Ising model, Monte Carlo calculation became a useful method to study critical phenomena. The concept of finite size scaling of Monte Carlo data has been applied to obtain critical exponents of various systems in various dimensions [31].

A material system may exist in different phases. An ordinary (classical) fluid may exist in the solid, the liquid and the gas phase. The distinction between the phases can be made by the qualitative difference among the phases. A phase transition of a system lies at the border between phases. There are two main kinds of phase transitions, first order phase transitions (discontinuous) and second order (continuous) phase transitions. A first order phase transition involves a discontinuous change in some intensive thermodynamic quantity, i.e. in the first derivative of a thermodynamic potential. When there is a phase transition between two phases, the two different thermodynamic phases can coexist under the same external thermodynamic conditions. That means the free energies per particle of the different phases in equilibrium are the same.
In a ferromagnetic system, there are two thermodynamic phases, the paramagnetic phase and the ferromagnetic phase. The temperature at which the system changes from one phase to another is the critical temperature, $T_c$, or say the critical point. There is a considerable randomness to the arrangement of spins in a paramagnetic phase, but all the spins in a ferromagnetic phase tend to point in one direction. At criticality, a situation intermediate between the ferromagnet and the paramagnet exists, and the spins are correlated over large distances and long times although some randomness still exists.

In a magnetic system, the correlation length $\xi$ describes the distance over which a specific thermodynamic variable in the system is correlated. Above $T_c$, spins over a short distance are correlated with one another, this is called short-range order. As $T_c$ is approached from above, the correlation distance becomes larger and larger. Below $T_c$, infinite-range correlations among spins exist, which means the system possesses long-range order. The pair correlation function describes how the spins are correlated within a system. It is defined as

$$\Gamma(r, T) = \langle (s_i - \langle s \rangle) (s_j - \langle s \rangle) \rangle.$$  \hspace{1cm} (2.1)

Here, $s_i$ and $s_j$ are the spins at site $i$ and $j$, and $r$ is the distance between $i$ and $j$.

In a magnet, if a first order phase transition occurs, the direction of magnetization might suddenly change so that the magnetization density vector will change discontinuously. Here, the magnetization, or the magnetization density, $M(T)$, which is the discontinuous thermodynamic variable, is called the order parameter. If the magnetization density vector changes continuously with $T$, it is a second order phase transition. For a
second order transition the order parameter vanishes as \( M \sim (T_c - T)^\beta \) as \( T \) approaches \( T_c \) from below. Here \( \beta \) is the critical exponent associated with the order parameter. In zero magnetic field, the magnetization is the spontaneous magnetization for \( T < T_c \). Far below the critical temperature, \( T_c \), the magnetic system has long-range order, and the spontaneous magnetization density is not zero. If the temperature is increased from far below \( T_c \) and approaches \( T_c \), the spontaneous magnetization approaches zero. Above the critical temperature, \( T_c \), long-range order vanishes and the spontaneous magnetization \( M(T) \) is zero.

### 2.1.1 Critical Point and Phase transitions in Ising models

Many theories and models have been made to understand magnetic transitions since the early years of the twentieth century. Some of the models are constructed by assuming the magnetic moments are localized on fixed lattice sites and interact with each other through pairwise interactions. Two models with particular forms of the interactions are studied a lot today, the Ising model \[23\] and the Heisenberg model \[18\]. In the Ising model, the magnetic moments are assumed to be classical, each spin can only point in two opposite directions. The Heisenberg model treats the magnetic moments as being related to quantum-mechanical three-component spin operators, and assumes that the energy is proportional to the scalar product of these operators. The Ising model in one dimension is quite easy to solve exactly \[63\] \[21\], but neither the Ising nor the Heisenberg model has yielded as yet to an exact solution for a three-dimensional lattice. There is an exact solution
for the zero-field Ising model on a square lattice [63] and other regular planar lattices. In this dissertation, the Ising models on different lattices derived from the one-dimensional linear lattice or the two-dimensional square lattice are studied.

The Ising model attempts to capture the structure of a physical magnetic substance. In the Ising model, the system being considered is an array of \( N \) fixed lattice sites. The lattice is \( n \)-dimensional \((n = 1, 2, 3, 4)\) with particular boundary conditions. Periodic boundary conditions are described here or utilized in the simulation. A lattice is said to be with periodic boundary conditions if the last site in a dimension is assumed to be connected with the first site in the same dimension. The geometrical structure of the lattice for the Ising model may be square, triangular, and hexagonal in the two-dimensional case and the simple cubic, the body-centered cubic and the face-centered cubic in the three-dimensional case. It can be simple hypercubic or body-centered hypercubic in the four-dimensional case. A spin associated with a lattice site has a variable \( s_i \) \((i = 1, 2, 3..., N)\) which is a number that is either 1 or \(-1\) for each of the \( N \) spins in the lattice. Here \( s_i \) can be viewed as the sign of the projection of a quantum spin \( \frac{1}{2} \) particle onto the \( z \) axis, taken to be the axis along which an applied field is applied.

The Hamiltonian of the Ising model is

\[
\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} s_i s_j - H \sum_i s_i, \tag{2.2}
\]
where \( s_i = \pm 1 \), and the first summation extends over all spin pairs \( \langle ij \rangle \). The interaction coefficient is \( J_{ij} \) and \( H \) is the external magnetic field. If the interaction only exists between the nearest neighbor pairs and with the same intensity, then

\[
\mathcal{H} = - J \sum_{\langle ij \rangle} s_i s_j - H \sum_i s_i. \tag{2.3}
\]

Here the first summation is over all nearest neighbor spin pairs \( \langle ij \rangle \). An interaction coefficient \( J > 0 \) corresponds to a ferromagnet and \( J < 0 \) corresponds to an antiferromagnet.

The partition function is defined by

\[
Z(T, H) = \sum_{\{s_i\}} e^{-\frac{\mathcal{H}}{k_B T}} = \sum_{\{s_i\}} e^{-\beta \mathcal{H}} \tag{2.4}
\]

where the summation is over all \( s_i \) configurations. For the Ising model, there are \( 2^N \) configurations. Here \( k_B \) is Boltzmann’s constant and \( \beta = \frac{1}{k_B T} \).

In a magnetic system, there are two pairs of reciprocal variables, the entropy \( S \) and the temperature \( T \) form one pair, while the external magnetic field \( H \) and the magnetization \( M \) form the other. Consequently there are four state functions that can be used to describe the system state. They are the Gibbs potential \( G(T, H) \), the internal energy \( U(S, M) \), the enthalpy \( F(S, H) \), and the Helmholtz potential \( A(T, M) \). The Gibbs potential is defined as

\[
G(T, H) = -k_B T \ln Z(T, H). \tag{2.5}
\]

Here \( \ln Z(T, H) \) is the logarithm of the partition function \( Z(T, H) \) defined in Equation (2.4). The four state functions are equivalent to one another, and can be obtained
from one another by their relations. For example, once the Gibbs potential is defined, the other three can be defined as,

\[ A(T, M) = G(T, H) + HM \] (2.6)

\[ F(S, H) = G(T, H) + TS \] (2.7)

\[ U(S, M) = -k_B T^2 \frac{\partial}{\partial T} \left( \frac{G}{k_B T} \right). \] (2.8)

One variable from each pair of \((S, T), (H, M)\) can be derived from the state functions. For example, the magnetization \(M\) can be derived from \(G\) and expressed as a function in terms of \(T\) and \(H\),

\[ M(T, H) = -\left( \frac{\partial G}{\partial H} \right)_T. \] (2.9)

The magnetization \(M\) is used as the order parameters in magnetic phase transitions, so the relationship of \(M\) with respect to \(T\) and \(H\) is important. The relation of \(M\) with \(H\) at different \(T\) is drawn in Figure 2.1, from which it can be seen that at \(H = 0\), \(M\) is zero above \(T_c\) and non-zero below \(T_c\). The zero-field value \(M(T, 0)\) is the spontaneous magnetization, it is also written as \(M_0(T)\). If it is nonzero, the system is ferromagnetic.

It is useful to define some magnetic response functions for analyzing magnetic systems. A few of them are the isothermal susceptibility \(\chi_T\), the adiabatic susceptibility \(\chi_S\), the isomagnetic field specific heat \(C_H\), and the isomagnetization specific heat \(C_M\). The following are their definitions,

\[ C_H \equiv T \left( \frac{\partial S}{\partial T} \right)_H = -T \left( \frac{\partial^2 G}{\partial T^2} \right)_H \] (2.10)
Figure 2.1 The magnetization $M$ as a function of $H$ and $T$, the curve is drawn for the mean field model

Note: Here, $M_0$ is the saturation magnetization. In (a), it is seen that $M(H)$ is discontinuous at $H = 0$ for $T < T_c$. In (b), $H = 0$, $M(T) = 0$ for $T > T_c$, and $M(T)$ is given by the discontinuity in (a) for $T < T_c$.

\[ C_M \equiv T \left( \frac{\partial S}{\partial T} \right)_M = -T \left( \frac{\partial^2 A}{\partial T^2} \right)_M \]  \hfill (2.11)  
\[ \chi_T \equiv \left( \frac{\partial M}{\partial H} \right)_T = - \left( \frac{\partial^2 G}{\partial H^2} \right)_T \]  \hfill (2.12)  
\[ \chi_S \equiv \left( \frac{\partial M}{\partial H} \right)_S = - \left( \frac{\partial^2 F}{\partial H^2} \right)_S \].  \hfill (2.13)  

These response functions have the following relations,

\[ \chi_T (C_H - C_M) = T \alpha_H^2 \]  \hfill (2.14)  
\[ C_H (\chi_T - \chi_S) = T \alpha_H^2 \]  \hfill (2.15)  
\[ \frac{C_H}{C_M} = \frac{\chi_T}{\chi_S} \]  \hfill (2.16)  
\[ C_H - C_M = T \alpha_M^2 \chi_T \].  \hfill (2.17)
Here $\alpha_H$ and $\alpha_M$ are defined as,

$$\alpha_H \equiv \left( \frac{\partial M}{\partial T} \right)_H$$  \hspace{1cm} (2.18)

$$\alpha_M \equiv \left( \frac{\partial H}{\partial T} \right)_M = -\frac{\alpha_H}{\chi_T}.$$  \hspace{1cm} (2.19)

In zero field ($H = 0$), from Equation (2.2), Equation (2.4), Equation (2.5), Equation (2.9), Equation (2.10) and Equation (2.12) to obtain the free energy $E$, the specific heat $C_H$, the magnetization $M$ and the susceptibility $\chi$.

$$E = \langle H \rangle = \langle -J \sum_{ij} s_is_j \rangle$$  \hspace{1cm} (2.20)

$$C_H = \beta^2 \left( \langle E^2 \rangle - \langle E \rangle^2 \right) = \frac{\langle E^2 \rangle - \langle E \rangle^2}{(k_BT)^2}$$  \hspace{1cm} (2.21)

$$M = \langle \sum_i s_i \rangle$$  \hspace{1cm} (2.22)

$$\chi_T = \beta \left( \langle M^2 \rangle - \langle M \rangle^2 \right) = \frac{\langle M^2 \rangle - \langle M \rangle^2}{k_BT}$$  \hspace{1cm} (2.23)

Here $\langle S \rangle$ is the statistic average of quantity $S$. It is theoretically defined as

$$\langle S \rangle = \frac{\sum_k S_ke^{-\frac{M}{k_BT}}}{\sum e^{-\frac{M}{k_BT}}}$$  \hspace{1cm} (2.24)

where $S_k$ is the value of $S$ at state $k$ and the summation extends over all thermodynamical states. In experiment or simulation, $\langle S \rangle$ is calculated from $N_{trial}$ trials, and

$$\langle S \rangle = \frac{1}{N_{trial}} \sum_k^{N_{trial}} S_k,$$  \hspace{1cm} (2.25)

where $S_k$ is the value of $S$ at the $k$th trial.
2.1.2 The Critical-Point Exponents $\beta$, $\gamma$, $\alpha$, $\delta$, $\nu$, $\eta$

The study of critical phenomena of second order transitions has come to focus more and more on the values of a set of indices, called critical-point exponents, which describe the behavior of the interest quantities near the critical point.

Consider the scaled temperature $t$,

$$t \equiv \frac{T - T_c}{T_c} = \frac{T}{T_c} - 1$$

(2.26)

which is a dimensionless variable to measure the distance from the critical temperature $T_c$. A general function $f(t)$ can be expressed as

$$f(t) = At^x(1 + Bt^y + \cdots) \quad [y > 0]$$

(2.27)

if $f(t)$ is positive and continuous for sufficiently small, positive values of $t$. The limit

$$\lambda \equiv \lim_{t \to 0} \frac{\ln f(t)}{\ln t} = x$$

(2.28)

is called the critical-point exponent associated with the function $f(t)$. Generally, it is written as $f(t) \sim t^\lambda$.

In a magnetic system, magnetic variables and response functions can be expanded in terms of $t$ in the form of Equation (2.27). Here are the expansions of the spontaneous magnetization $M_0(T)$, the isothermal susceptibility $\chi_T$, the correlation length $\xi$, and the isomagnetic field specific heat $C_H$ in terms of $t$ [63].

$$\frac{M_0(T)}{M_0(0)} = A_\beta(-t)^\beta(1 + \cdots)$$

(2.29)
\[
\chi_T / \chi_T^0 = \begin{cases} 
A_\gamma (-t)^{-\gamma'}(1 + \cdots) & [T < T_c, H = 0] \\
A_\gamma t^{-\gamma}(1 + \cdots) & [T > T_c, H = 0]
\end{cases}
\] (2.30)

\[
C_H = \begin{cases} 
A_{CH}(-t)^{-\alpha'}(1 + \cdots) & [T < T_c, H = 0] \\
A_{CH}t^{-\alpha}(1 + \cdots) & [T > T_c, H = 0]
\end{cases}
\] (2.31)

\[
\xi = \begin{cases} 
A_\xi (-t)^{-\nu'}(1 + \cdots) & [T < T_c, H = 0] \\
A_\xi t^{-\nu}(1 + \cdots) & [T > T_c, H = 0]
\end{cases}
\] (2.32)

Here \(M_0(0)\) is a normalization constant, \(\chi_T^0\) is the susceptibility of a system of non-interacting magnetic moments (paramagnet) evaluated at the critical point. The exponents \(\beta, \gamma', \gamma, \alpha', \alpha, \nu\) are called zero-field critical exponents. The critical amplitudes are \(A_\beta, A_\gamma, A_{CH}, A_{CH'}, A_\xi, A_\xi\). At the critical point, the relation between the magnetic field \(H_c\) and the magnetization \(M_H\) is,

\[
\frac{H_c^0}{H_c} = A_H \left| \frac{M_H(T = T_c)}{M_0(T = 0)} \right|^\delta \quad [T = T_c],
\] (2.33)

Here \(H_c^0 \equiv \frac{kT_c}{m_0}\), where \(m_0\) is the magnetic moment per spin at the critical point. The exponent \(\delta\) is called the critical isothermal exponent and the corresponding critical amplitude is \(A_H\). In the case of \(H = 0\) and \(T = T_c\), the pair-correlation function \(\Gamma(r; t)\) is

\[
\Gamma(r) \sim \frac{1}{r^{\eta+2}} \quad [T = T_c, H = 0],
\] (2.34)

where \(\eta\) is called the pair-correlation exponent.

In summary, the definitions of the above critical exponents are shown in Table 2.1 [63] and the values for Mean-field model and some Ising models are shown in Table 2.2 [27] [63].
### Table 2.1 Summary of definitions of critical-point exponents.

<table>
<thead>
<tr>
<th>exponent</th>
<th>definition</th>
<th>( t \equiv T/T_c - 1 )</th>
<th>( d = \text{dimension} )</th>
<th>( t )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha' )</td>
<td>( C_H \sim (-t)^{-\alpha'} )</td>
<td>( &lt; 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( C_H \sim t^{-\alpha} )</td>
<td>( &gt; 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( M \sim (-t)^\beta )</td>
<td>( &lt; 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \gamma' )</td>
<td>( \chi_T \sim (-t)^{-\gamma'} )</td>
<td>( &lt; 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>( \chi_T \sim t^{-\gamma} )</td>
<td>( &gt; 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \delta )</td>
<td>( H \sim</td>
<td>M</td>
<td>^d \text{sgn}(M) )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \nu' )</td>
<td>( \xi \sim (-t)^{-\nu'} )</td>
<td>( &lt; 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \nu )</td>
<td>( \xi \sim t^{-\nu} )</td>
<td>( &gt; 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \eta )</td>
<td>( \Gamma(r) \sim</td>
<td>r</td>
<td>^{-(d-2+\eta)} )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

### Table 2.2 Values of critical-point exponents for magnetic Ising systems.

<table>
<thead>
<tr>
<th>dimension</th>
<th>( T &lt; T_c )</th>
<th>( T = T_c )</th>
<th>( T &gt; T_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha' )</td>
<td>( \beta )</td>
<td>( \gamma' )</td>
</tr>
<tr>
<td>( d = 2 )</td>
<td>0(log)</td>
<td>1/8</td>
<td>1</td>
</tr>
<tr>
<td>( d = 3 )</td>
<td>( \sim 1/8 )</td>
<td>( \sim 5/16 )</td>
<td>( \sim 5/4 )</td>
</tr>
<tr>
<td>( d = 4 )</td>
<td>0(discont.)</td>
<td>1/3</td>
<td>1</td>
</tr>
<tr>
<td>Mean field</td>
<td>0(discont.)</td>
<td>1</td>
<td>1/3</td>
</tr>
</tbody>
</table>
2.2 The mean field theory of two-level spin systems

The first mean field model for a ferromagnet was proposed by Pierre Weiss in 1907 [66]. In his model, he assumed the interaction between the spins can be considered to be that of the spins interacting with a molecular field proportional to the average magnetization of the whole system. It is convenient to construct a magnetic model in which each spin (magnetic moment) interacts with every other spin with an equal strength. Thus the spins in a magnetic system interact with a mean magnetic field of the whole system. Through this mean field, the state (or properties) of one spin can communicate with others. It is clear that such a mean field model can be used to study critical phenomena.

In this section, the mean field theory of two-level ferromagnetic spin systems will be discussed.

2.2.1 Magnetic system and Critical temperature

In a two-level magnetic system, each spin has two magnetic moment values, ±1. The following derivation follows that of [63], except that here $s \pm 1$ rather than $\pm \frac{1}{2}$ is used. The magnetization is

$$M = \sum_{i} m_i.$$  \tag{2.35}
Here $m_i = \pm 1$. The total number of spins in the system is $N$. Every spin will cause a magnetic field $H_m$ seen by the other spins in addition to the external magnetic field $H$. If $H_m$ is assumed to be proportional to the magnetization $M$, then the effective field is,

$$H_{\text{eff}} = H + \lambda M(T, H). \quad (2.36)$$

Here $\lambda$ is a parameter. The Hamiltonian is

$$\mathcal{H} = -H_{\text{eff}} M = -\sum_{i=1}^{N} m_i H_{\text{eff}} = -\sum_{i=1}^{N} m_i (H + \lambda M). \quad (2.37)$$

Refer to Equation (2.4), the partition function is

$$Z(T, H) = \sum_{m_1} \sum_{m_2} \ldots \sum_{m_N} e^{m_i (H + \lambda M) \beta} = \prod_{i=1}^{N} \{ \sum_{m_i = -1}^{1} e^{x m_i} \}$$

$$= \prod_{i=1}^{N} 2 \cosh x = 2^N \cosh^N x. \quad (2.38)$$

Here

$$x = \frac{H + \lambda M}{k_B T} = (H + \lambda M) \beta. \quad (2.39)$$

From Equation (2.5) the Gibbs potential is

$$G(T, H) = -k_B T \ln Z(T, H) = -k_B T N \left[ \ln 2 + \ln(\cosh x) \right]. \quad (2.40)$$

From Equation (2.9), the magnetization is

$$M(T, H) = -\left( \frac{\partial G}{\partial H} \right)_T = N \tanh x. \quad (2.41)$$

If $M_0$ is defined as the value of $M$ at zero temperature and zero field, then

$$M_0 = M(T = 0, H = 0) = N, \quad (2.42)$$
so

\[ M(T, H) = M_0 \tanh x = M_0 \tanh \left( \frac{H + \lambda M}{k_B T} \right) = M_0 \tanh((H + \lambda M)\beta). \quad (2.43) \]

Thus Equation (2.43) defines the relationship between \( M \) and \( H \). For large \( T \), the equation has only one solution, \( M = 0 \). If \( T \) is small enough, the equation has three solutions.

![Graphical solutions of \( M = M_0 \tanh \left( \frac{H + \lambda M}{k_B T} \right) \) for small \( T \)](image)

Figure 2.2 shows how the \( y = M \) line and the \( y = M_0 \tanh((H + \lambda M)/k_B T) \) curve cross at small \( T \). From the figure, it can be seen if Equation (2.43) has \( M \neq 0 \) solutions, the slope of the curve at \( M = 0 \) must be larger than 1. The slope of the right side of Equation (2.43) at \( M = 0 \) is

\[ M_0 \beta \lambda = \frac{M_0 \lambda}{k_B T}. \quad (2.44) \]
so to have more than one solution

\[ 1 < \frac{M_0 \lambda}{k_B T} \quad (2.45) \]

\[ T < \frac{\lambda M_0}{k_B} \quad (2.46) \]

This leads to the critical temperature \( T_c \) of mean field theory for a spin \( s = \pm 1 \) system,

\[ T_c = \frac{\lambda M_0}{k_B} \quad (2.47) \]

Hence \( T_c \) is proportional to mean field parameter \( \lambda \). When \( \lambda \) goes to zero, \( T_c \) goes to zero.

The system is ferromagnetic \( (M > 0) \) for \( T < T_c \) and paramagnetic \( (M = 0) \) for \( T > T_c \).

### 2.2.2 Critical-point exponents

For easy discussion, define the reduced magnetization \( \sigma \),

\[ \sigma \equiv \frac{M}{M_0} = \frac{M(T, H)}{M(0, 0)} = \tanh \left( \frac{H + \lambda M}{k_B T} \right) \]

\[ = \tanh((H + \lambda M) \beta) = \tanh x. \quad (2.48) \]

In zero field \( (H = 0) \), using Equation (2.47), from the above equation, it can be obtained,

\[ \frac{M}{M_0} = \tanh \left( \frac{M T_c}{M_0 T} \right) \quad (2.49) \]

This is the relation between the order parameter and the temperature in mean field theory.

Define the reduced temperature \( \tilde{T} \) and the reduced magnetic field \( h \),

\[ \tilde{T} \equiv \frac{T}{T_c} \quad (2.50) \]

\[ h = \tanh(\beta H), \quad (2.51) \]
then from Equation (2.43),

\[ \sigma = \tanh \left( \beta H + \frac{\sigma}{T} \right) = \frac{h + \tanh \frac{\sigma}{T}}{1 - h \tanh \frac{\sigma}{T}}. \tag{2.52} \]

For small \( x \),

\[ \tanh x = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 + \cdots. \tag{2.53} \]

Using Equation (2.53) expand \( \tanh \left( \frac{\sigma}{T} \right) \) near \( H = 0 \), and substitute into Equation (2.52),

\[ h = \sigma \left( 1 - \frac{1}{T} \right) + \sigma^3 \left\{ \frac{1}{3T} + \left( 1 - \frac{1}{T} \right) \frac{1}{T} \right\} + O(\sigma^5). \tag{2.54} \]

Equation (2.54) is used to derive critical-point exponents in the mean field approximation.

In Equation (2.54), set \( \tilde{T} = 1 \), then for small \( M \) and \( H \), the \( M - H \) isothermal relation is reached,

\[ H \cong k_B T_c \frac{\sigma^3}{3} = \frac{k_B T_c}{3 M^3(0,0)} M^3(T, H) \tag{2.55} \]

thus

\[ \delta = 3. \tag{2.56} \]

At zero magnetic field \( (H = 0) \), from Equation (2.51) \( h = 0 \). Let \( h = 0 \) in Equation (2.54), then \( \sigma \) is given by

\[ \sigma^2 \cong 3 \left( \frac{T}{T_c} \right)^2 \left( \frac{T_c - T}{T_c} \right) \tag{2.57} \]

\[ \sigma \cong \sqrt{3} \frac{T}{T_c} \left( \frac{T_c - T}{T_c} \right)^{\frac{1}{2}}, \tag{2.58} \]

thus

\[ \beta = \frac{1}{2}. \tag{2.59} \]
It can be obtained from Equation (2.12), Equation (2.9), Equation (2.51), and Equation (2.42) that the zero-field isothermal susceptibility is

\[
\chi_{T,H=0} \equiv \left( \frac{\partial M}{\partial H} \right)_{T,H=0} = \frac{1}{k_B T} \left( \frac{\partial \sigma}{\partial T} \right)_{T,H=0} \left( \frac{\partial h}{\partial T} \right)_{T,H=0} = \frac{N}{k_B T} \left( \frac{\partial \sigma}{\partial H} \right)_{T,H=0}.
\] (2.60)

From the derivation of Equation (2.54)

\[
1 = \left( \frac{\partial \sigma}{\partial h} \right) \left\{ \left( 1 - \frac{1}{T} \right) + \sigma^2 \left( \frac{1}{T} \right) + \sigma^4 \left( \frac{1}{T} \right) \right\}^{-1} \] (2.61)

thus

\[
\chi_{T,H=0} = \frac{N}{k_B T} \left\{ \left( 1 - \frac{1}{T} \right) + \sigma^2 \left( \frac{1}{T} \right) + \sigma^4 \left( \frac{1}{T} \right) \right\}^{-1}. \] (2.62)

For \( T \geq T_c \), \( \sigma = 0 \) if \( H = 0 \),

\[
\chi_T = \frac{N}{k_B T} \left( \frac{T}{T - T_c} \right) \approx \frac{N}{k_B T_c} t^{-1}.
\] (2.63)

Hence

\[
\gamma = 1
\] (2.64)

with the coefficient \( \frac{N}{k_B T_c} \). For \( T \leq T_c \), from Equation (2.57)

\[
\sigma^2 \approx -3T^2 t
\] (2.65)

so

\[
\chi_T \approx \frac{N}{k_B T} \left\{ \left( 1 - \frac{1}{T} \right) - 3t \right\}^{-1} = \frac{N}{2k_B T_c} (-t)^{-1}
\] (2.66)

thus

\[
\gamma t = 1.
\] (2.67)
The coefficient is \( \frac{N}{2k_B T_c} \), which is one-half of the value for \( T \geq T_c \).

From Equation (2.40) and Equation (2.10),

\[
C_H \approx -T \frac{\partial^2 G}{\partial T^2} = \frac{N(H + \lambda M)^2}{k_B T^2} (1 - \tanh x).
\] (2.68)

For \( T > T_c \), \( M = 0 \) if \( H = 0 \), then

\[
C_H = 0
\] (2.69)

so

\[
\alpha = 0.
\] (2.70)

For \( T \leq T_c \), \( H = 0 \), using Equation (2.47), Equation (2.52), Equation (2.50), Equation (2.53), Equation (2.54)

\[
C_H = k_B N \left( \frac{\sigma}{T} \right)^2 \left( 1 - \tanh^2 \left( \frac{\sigma}{T} \right) \right)
\] (2.71)

\[
= 3k_B N + \circ \left( 1 - \frac{T}{T_c} \right)
\] (2.72)

so if \( T \to T_c^- \),

\[
C_H = 3k_B N \quad \text{(2.73)}
\]

\[
\alpha t = 0. \quad \text{(2.74)}
\]

There is a jump in \( C_H \) at \( T_c \) from \( 3k_B T \) to 0 when \( T \) goes from below \( T_c \) to above \( T_c \). This is consistent with a spin \((-\frac{1}{2}, \frac{1}{2})\) system, where \( C_H = \frac{3k_B N}{2} \) below \( T_c \) [63].
2.3 Landau’s classic theory of critical-point exponents

Landau introduced a general form of a thermodynamic potential near the critical point. At the critical-point, the isothermal susceptibility of a magnetic system is expected to be infinite and the specific heat to be finite. Thus a convergent power series expansion at the critical point may not lead to valuable results [63]. Landau pointed out that the power series may contain singularity coefficients at higher order, thus the low-order coefficients can be used to predict critical-point exponents.

2.3.1 Basic assumption of Landau’s classic theory

Since the Helmholtz potential is a function of $M$, it is convenient to use the Helmholtz potential expansion to discuss critical phenomena. At $T = T_c$ and $H = 0$,

$$A(T, M) = \sum_{j=0,2\cdots}^{\infty} L_j(T)M^j = L_0(T) + L_2(T)M^2 + L_4(T)M^4 + \cdots. \quad (2.75)$$

Here $L_j(T) = 0$ for $j = 2n + 1, \quad n = 0, 1, 2 \cdots$, since $A(T, M)$ is an even function of $M$. The coefficients $L_j(T)$ are functions of $T$, and so can be expanded about $T = T_c$,

$$L_j(T) = \sum_{k=0}^{\infty} l_{jk}(T - T_c)^k$$

$$= l_{j0} + l_{j1}(T - T_c) + l_{j2}(T - T_c)^2 + \cdots \quad (2.76)$$

so

$$H = H(T, M) = \left( \frac{\partial A}{\partial M} \right)_T = \sum_{j=2,4\cdots}^{\infty} jL_j(T)M^{j-1}$$

$$= 2L_2(T)M + 4L_4(T)M^3 + \cdots \quad (2.77)$$
\[ \chi_T^{-1} = \left( \frac{\partial^2 A}{\partial M^2} \right)_T = \sum_{j=2,4,\ldots}^{\infty} j(j-1)L_j(T)M^{j-2} \]

\[ = 2L_2(T) + 12L_4(T)M^2 + \ldots \]

(2.78)

For \( T \to T_c^+ \), \( M^4 \) is small, the zero-field susceptibility is supposed to approach infinity, so the first term of Equation (2.78) is important

\[ \chi_T^{-1}(T,0) = \left( \frac{\partial^2 A}{\partial M^2} \right)_T = 2L_2(T) + \ldots \]

\[ = 2(l_{20} + l_{21}(T - T_c) + l_{22}(T - T_c)^2 + \ldots) \]

(2.79)

and \( l_{20} \) must be zero.

There is another restriction to the Helmhotz potential that is discussed by some authors [63], but it is complicated and not directly relevant to this dissertation, and will not be included here.

### 2.3.2 Critical-point prediction of the Landau’s classic theory

In the neighborhood of the critical point \( T_c \), \( H = 0 \) and \( M \) is small, from Equation (2.77),

\[ 0 = 2\{l_{21}(T - T_c) + \ldots\} + 4M^2\{l_{40} + l_{41}(T - T_c) + \ldots\}. \]

(2.80)

Omit higher-order terms to obtain

\[ M = \left( \frac{l_{21}}{2l_{40}} \right)^{\frac{1}{2}} (T - T_c)^{\frac{1}{2}}. \]

(2.81)

Hence \( \beta = \frac{1}{2} \) is obtained, which is the same as in mean field theory.
From Equation (2.78) and Equation (2.76)

\[
\chi_T^{-1} = 2\{l_{21}(T - T_c) + \cdots\} + 12M^2\{l_{40} + l_{41}(T - T_c) + \cdots\}.
\]  
(2.82)

For \(T > T_c\), \(M = 0\) if \(H = 0\), the zero-field susceptibility is

\[
\chi_T^{-1}(T, 0) = 2l_{21}(T - T_c)
\]  
(2.83)

thus

\[
\gamma = 1.
\]  
(2.84)

For \(T \leq T_c\), the zero-field magnetization is not zero. Substitute \(M\) from Equation (2.81) into Equation (2.82), to obtain

\[
\chi_T^{-1}(T, 0) = 4l_{21}(T - T_c) + \cdots
\]  
(2.85)

thus

\[
\gamma' = 1 = \gamma.
\]  
(2.86)

Compare Equation (2.83) to Equation (2.85), when \(T_c\) is approached from below \(T_c\), the susceptibility rises half as fast as from above \(T_c\). This result is also consistent with the result of the mean-field theory.

From Equation (2.77), and \(l_{20} = 0\), \(T = T_c\), the isothermal equation \(H\) of \(M\) becomes

\[
H(T_c, M) = 4l_{40}M^3 + \cdots
\]  
(2.87)

thus

\[
\delta = 3.
\]  
(2.88)
For \( T > T_c \), \( M = 0 \) if \( H = 0 \), from Equation (2.11), Equation (2.75), Equation (2.76), the zero-field specific heat is

\[
C_H = C_M = -2l_{02}T_c - (6l_{03}T_c + 2l_{02})(T - T_c) + \cdots + \mathcal{O}((T - T_c)^2). \tag{2.89}
\]

Thus

\[
\alpha = 0. \tag{2.90}
\]

While for \( T \leq T_c \), from Equation (2.11), Equation (2.75), Equation (2.76), and Equation (2.81)

\[
C_M = -2l_{02}T_c - \left( \left( 6l_{03} - \frac{l_{21}l_{22}}{l_{40}} \right) T_c + 2l_{02} \right) (T - T_c) + \cdots + \mathcal{O}((T - T_c)^2), \tag{2.91}
\]

and from Equation (2.19), Equation (2.75), Equation (2.76), for small \( M \),

\[
\alpha_M \asymp 2l_{21}M. \tag{2.92}
\]

Then from Equation (2.17), Equation (2.81), and Equation (2.85), one has

\[
C_H - C_M = T \left( \frac{l_{21}^2}{2l_{40}} \right) (1 + \mathcal{O}(T - T_c)) \tag{2.93}
\]

and

\[
C_H = \left( \frac{l_{21}^2}{2l_{40}} - 2l_{02} \right) T_c + \left\{ \frac{l_{21}^2}{2l_{40}} - \left[ \left( 6l_{03} - \frac{l_{21}l_{22}}{l_{40}} \right) T_c + 2l_{02} \right] \right\} (T - T_c) + \mathcal{O}((T - T_c)^2). \tag{2.94}
\]

Thus

\[
\alpha t = 0. \tag{2.95}
\]
2.4 Scaling of thermodynamic functions

The thermodynamic properties of a system are related to system size. Hence the corresponding scaling function is important in the study of thermodynamic systems.

2.4.1 Homogeneous function

For a function of \( n \) variables, \( f(x_1, x_2, x_3, \cdots, x_n) \), if for all values of the parameter \( \lambda > 0 \), and \( n \) parameters \( y_1, y_2, y_3, \cdots, y_n \),

\[
f(\lambda y_1 x_1, \lambda y_2 x_2, \lambda y_3 x_3, \cdots, \lambda y_n x_n) = \lambda f(x_1, x_2, x_3, \cdots, x_n),
\]

(2.96)

the function \( f(x_1, x_2, x_3, \cdots, x_n) \) is called a generalized homogeneous function. For convenience, Equation (2.96) sometimes is written in its equivalent form

\[
f(\lambda y_1 x_1, \lambda y_2 x_2, \lambda y_3 x_3, \cdots, \lambda y_n x_n) = \lambda^p f(x_1, x_2, x_3, \cdots, x_n)
\]

(2.97)

or

\[
f(x_1, x_2, x_3, \cdots, x_n) = \lambda^{-p} f(\lambda y_1 x_1, \lambda y_2 x_2, \lambda y_3 x_3, \cdots, \lambda y_n x_n).
\]

(2.98)

Set \( \lambda y_1 |x_1| = 1 \), then \( \lambda = |x_1|^{-\frac{1}{y_1}} \), so

\[
f(x_1, x_2, x_3, \cdots, x_n) = |x_1|^{\frac{p}{y_1}} f\left(sign(x_1), x_2 |x_1|^{-\frac{y_2}{y_1}}, x_3 |x_1|^{-\frac{y_3}{y_1}}, \cdots, x_n |x_1|^{-\frac{y_n}{y_1}}\right)
\]

(2.99)

In the two variables case, \( n = 2 \),

\[
f(x_1, x_2) = |x_1|^{\frac{p}{y_1}} f\left(sign(x_1), x_2 |x_1|^{-\frac{y_2}{y_1}}\right)
\]

(2.100)
or its equivalent form,
\[ f(\lambda^y_1 x_1, \lambda^y_2 x_2) = \lambda f(x_1, x_2). \] (2.101)

### 2.4.2 Static scaling of thermodynamic functions in zero field

It can be shown that if one of the four thermodynamic potentials, \( U(S, M) \), \( F(S, H) \), \( G(T, H) \) and \( A(T, M) \), is a generalized homogeneous function, then the other three are all generalized homogeneous functions [63]. It is easy to obtain critical point exponents and their relations from the Gibbs potential, so the Gibbs potential is the most used one in static scaling discussions.

When the reduced temperature \( t = \frac{T - T_c}{T_c} \) is used instead of \( T \), the Gibbs potential is a function of the reduced temperature \( t \) and the magnetic field \( H \). In a static scaling analysis, the Gibbs potential is assumed to be a generalized homogeneous function. From Equation (2.101), the Gibbs potential should scale as

\[ G(t, H) = \lambda^{-1} G(\lambda^a_t t, \lambda^a_H H). \] (2.102)

Since all the response functions can be derived from the Gibbs potential, and the Gibbs potential scales with only 2 parameters \( a_t \) and \( a_H \), all of the critical-point exponents can be expressed in terms of \( a_t \) and \( a_H \). Details of the derivation can be found in [63].

From Equation (2.9) and Equation (2.102)

\[ \frac{\partial G(t, H)}{\partial H} = \lambda^{-1} \lambda^a_H \frac{\partial G(\lambda^a_t t, \lambda^a_H H)}{\partial (\lambda^a_H H)} \] (2.103)
\[ M(t, H) = \lambda^{\alpha_H^{-1}} M(\lambda^{\alpha_H t}, \lambda^\alpha H). \] (2.104)

Let \( \lambda = (-t)^{-\frac{1}{\alpha_H}} \) for \( T < T_c \), and using Equation (2.101)

\[ M(t, H) = (-t)^{-\frac{(\alpha_H - 1)}{\alpha}} M(-1, (-t)^{-\frac{\alpha_H}{\alpha}} H). \] (2.105)

In the case of \( H = 0 \), \( t \to 0^- \),

\[ M(t, 0) = (-t)^{-\frac{(\alpha_H - 1)}{\alpha}} M(-1, 0). \] (2.106)

From Equation (2.29), \( M(t, 0) \) should proportional to \((-t)^\beta\) when \( t \to 0^- \), so

\[ \beta = -\frac{\alpha_H - 1}{\alpha_t}. \] (2.107)

If \( t = 0 \) and \( H \to 0 \), let \( \lambda = H^{-\frac{1}{\alpha_H}} \), using Equation (2.101) and Equation (2.104)

\[ M(0, H) = H^{-\frac{(\alpha_H - 1)}{\alpha_H}} M(0, 1), \] (2.108)

compare to Equation (2.33)

\[ M(0, H) \sim H^{\frac{1}{\delta}} \] (2.109)

so,

\[ \delta = -\frac{\alpha_H}{\alpha_H - 1}. \] (2.110)

From Equation (2.107) and Equation (2.110),

\[ a_t = \frac{1}{\beta \delta + 1} \] (2.111)

and

\[ a_H = \delta \frac{1}{\delta + 1}. \] (2.112)
For the isothermal susceptibility, using the scaling Equation (2.102), differentiate the Gibbs potential \( G(t, H) \) two times with respect to \( H \) and then let \( H = 0 \). For \( T < T_c \), let 
\[
\lambda = (-t)^{-\frac{1}{2\alpha}}, 
\]
then
\[
\chi_T(t, 0) = (-t)^{-\frac{(2\alpha_H - 1)}{2\alpha}} \chi_T(-1, 0). 
\] (2.113)

Since \( t \to 0^- \),
\[
\chi_T(t, 0) \sim (-t)^{-\gamma'} 
\] (2.114)
so
\[
\gamma' = 2\alpha_H - 1 = \beta(\delta - 1). 
\] (2.115)

For \( T > T_c \), let \( \lambda = t^{-\frac{1}{2\alpha H}} \), to obtain
\[
\gamma = 2\alpha_H - 1 = \beta(\delta - 1) = \gamma'. 
\] (2.116)

For the specific heat, using the scaling Equation (2.102), differentiate the Gibbs potential \( G(t, H) \) two times with respect to \( t \) and then let \( H = 0 \) to obtain the \( C_H \) scaling relation, for \( T < T_c \),
\[
C_H(t, 0) = (-t)^{-\frac{(2\alpha_H - 1)}{2\alpha t}} C_H(-1, 0) 
\] (2.117)
and for \( T > T_c \),
\[
C_H(t, 0) = t^{-\frac{(2\alpha_H - 1)}{2\alpha t}} C_H(1, 0) 
\] (2.118)
so it follows that
\[
\alpha' = \alpha = 2 - \frac{1}{\alpha t}. 
\] (2.119)
Combine Equation (2.112), Equation (2.111), Equation (2.119), Equation (2.107), and Equation (2.116),
\[ \alpha + 2\beta + \gamma = 2 \] (2.120)
and
\[ \alpha' + 2\beta + \gamma' = 2. \] (2.121)

2.4.3 Finite Size Scaling

Monte Carlo (MC) simulation is an important approach to study critical phenomena. Since only finite lattices can be modeled in computers, more attention is paid to finite volume methods. The fact that thermodynamic quantity correlations and their changes are affected by the correlation length leads to the concept of Finite Size Scaling (FSS). Finite Size Scaling theory concerns estimating the proper relation between the volume size of a system and the critical point, associated with the critical exponents. FSS of MC simulation is one of the most effective techniques for the determination of critical quantities.

For a finite size ferromagnetic system, thermodynamic quantities are functions of the system volume $V$, or the total number of particles in the system $N$. A $d$-dimensional system is often characterized by a linear size, say $L$, which is defined as
\[ L = V^{\frac{1}{d}}. \] (2.122)

This doesn’t mean that each of the dimensions of the system is of the length $L$, although a lattice with all dimensions equal to $L$ is often used for convenience. In a simple lattice
system (each lattice cell only contains one particle), if the lattice cell spacing is $a$, the cell volume is $a^d$ and the lattice volume is $V$, the total number of particles in the volume is

$$N = \frac{V}{a^d} = \left(\frac{L}{a}\right)^d. \quad (2.123)$$

If $a$ is set to be 1, then

$$N = V = L^d. \quad (2.124)$$

In this dissertation $a = 1$ is used. Privman and Fisher introduced a scaled external magnetic field $h$ [59], with

$$H = h k_B T \quad (2.125)$$

and rewrote the Hamiltonian of the Ising model,

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - H a^d \sum_i s_i. \quad (2.126)$$

Then the density of the four potential (or reduced potential) can be written in the following form,

$$\phi(t, h, L) \equiv \frac{\Phi(T, H, L)}{k_B T V}. \quad (2.127)$$

For example, the Gibbs potential density $g$ is,

$$g(t, h, L) \equiv \frac{G(T, H, L)}{k_B T V}. \quad (2.128)$$

The density of response functions can also be expressed in terms of $t$, $h$, and $L$. The magnetization is

$$m(t, h, L) \equiv -\frac{1}{V} \left( \frac{\partial G(T, H, L)}{\partial H} \right)_T. \quad (2.129)$$
For convenience, $G(t, h, L)$ and $M(t, h, L)$ are used instead of $g(t, h, L)$ and $m(t, h, L)$ for the density of the corresponding magnetic quantities $G(T, H, L)$ and $M(T, H, L)$. Also, $\chi(t, h, L)$ and $C(t, h, L)$ represent the density of the susceptibility and the specific heat.

The Gibbs potential $G(t, h)$ and the pair correlation function $\Gamma(t, h)$ scale as [63]

$$G(t, h) = L^{-d}G(L^{y_h}t, L^{y_h}h)$$

and

$$\Gamma(r, t) = L^{2(y_h-d)}\Gamma(L^{-1}r, L^{y_t}t)$$

where the pair correlation is defined as in Equation (2.1).

There are relations for $y_h, y_t, d, \nu, \eta, \gamma, \alpha$ [63],

$$y_h = \frac{d + 2 - \eta}{2}$$  \hspace{1cm}  (2.132)

$$y_t = \frac{1}{\nu}$$  \hspace{1cm}  (2.133)

$$d\nu = 2 - \alpha$$  \hspace{1cm}  (2.134)

$$(2 - \eta)\nu = \gamma.$$  \hspace{1cm}  (2.135)

So the scaling of $G$ is

$$G(t, h) = L^{-d}G\left(L^{\frac{1}{\nu}}t, L^{\frac{d-\eta+2}{2}}h\right).$$

In zero field, $h = 0$ if $H = 0$,

$$G(t, 0) = L^{-d}G(L^{\frac{1}{\nu}}t, 0).$$  \hspace{1cm}  (2.137)

Let

$$\xi = A\xi t^{-\nu}$$

(2.138)
so
\[ t = A_\xi \xi^{-\frac{1}{2}}. \] (2.139)

Then
\[ G(t, 0) = L^{-d} G \left( L^{\frac{1}{2}} G_\xi \xi^{-\frac{1}{2}}, 0 \right) = L^{-d} G \left( \frac{L}{\xi}, 0 \right). \] (2.140)

Pelissetto and Vicari proposed the Helmholtz potential scaling relation [57],
\[ A(t, h, L) = L^{-d} A(L^{\eta} t, L^{\eta} h, 1). \] (2.141)

Since \( G(t, h, L) \) is linearly related to \( A(t, h, L) \), compare with Equation (2.136), the following scaling equations are hypothesized.
\[ A(t, h, L) = L^{-d} A(L^{\eta} t, L^{\frac{d-\eta+2}{2}} h, 1) = L^{-d} A \left( A_\xi f \left( \frac{\xi}{L} \right), L^{\frac{d-\eta+2}{2}} h, 1 \right) \] (2.142)
\[ G(t, h, L) = L^{-d} G(L^{\eta} t, L^{\frac{d-\eta+2}{2}} h, 1) = L^{-d} G \left( G_\xi f \left( \frac{\xi}{L} \right), L^{\frac{d-\eta+2}{2}} h, 1 \right). \] (2.143)

For zero field, \( h = 0 \), the above relations reduce to,
\[ A(t, L) = L^{-d} A \left( A_\xi f \left( \frac{\xi}{L} \right), 1 \right) \] (2.144)
and
\[ G(t, L) = L^{-d} G \left( G_\xi f \left( \frac{\xi}{L} \right), 1 \right). \] (2.145)

Here
\[ f \left( \frac{\xi}{L} \right) = \left( \frac{\xi}{L} \right)^{-\frac{1}{2}}. \] (2.146)

Pelissetto and Vicari also concluded that the FSS behavior of any thermodynamic quantity can be obtained from the above scaling relations. In zero field, \( H = 0 \) then \( h = 0 \),
any thermodynamic quantity $S(t, L)$ with $S_\infty \equiv S(t, \infty)$ behaves as $t^{-\sigma}$ for $t \to 0$, the following scaling relation holds [57],

$$S(t, L) = L^{\sigma \nu} \left[ f_{S,\xi_\infty} \left( \frac{\xi_\infty}{L} \right) + \mathcal{O} \left( L^{-\omega}, \xi_\infty^{-\omega} \right) \right]$$

$$= L^{\sigma \nu} \left[ f_{S,\xi} \left( \frac{\xi(t, L)}{L} \right) + \mathcal{O} \left( L^{-\omega}, \xi_\infty^{-\omega} \right) \right]$$

(2.147)

where $\xi_\infty$ is the correlation length of an infinite size system, $\omega$ is an exponent of an order that can be neglected, and $\xi(t, L)$ is the correlation length defined in a finite size lattice.

Thus the following scaling relations are obtained,

$$M(t, L) = L^{-\frac{\beta}{\nu}} f_{M,\xi_\infty} \left( \frac{\xi_\infty}{L} \right) = L^{-\frac{\beta}{\nu}} f_{M}(t L^{\frac{1}{\nu}})$$

(2.148)

$$\chi(t, L) = L^{\frac{\gamma}{\nu}} f_{\chi,\xi_\infty} \left( \frac{\xi_\infty}{L} \right) = L^{\frac{\gamma}{\nu}} f_{\chi}(t L^{\frac{1}{\nu}})$$

(2.149)

$$C(t, L) = L^{\frac{\alpha}{\nu}} f_{C,\xi_\infty} \left( \frac{\xi_\infty}{L} \right) = L^{\frac{\alpha}{\nu}} f_{C}(t L^{\frac{1}{\nu}}).$$

(2.150)

Here Equation (2.139), which is the relation for an infinite size lattice is used. Also, $f_{M}(x)$, $f_{\chi}(x)$ and $f_{C}(x)$ are the corresponding scaling functions. These results are the same as in reference [31]. Equation (2.148), Equation (2.149) and Equation (2.150) are used to determine $\frac{\beta}{\nu}$, $\frac{\gamma}{\nu}$ and $\frac{\alpha}{\nu}$. Figure 2.3 and Figure 2.4 are examples of the order parameter scaling and the susceptibility scaling for 2-dimensional square lattice Ising model. From these figures, it can be seen that the critical exponents are $\beta = \frac{1}{8}$, $\gamma = \frac{7}{4}$ and $\nu = 1$.

In Figure 2.3, the branch for $T < T_c$ approaches to a slope $\frac{1}{8}$ line as $t L^{\frac{1}{\nu}}$ becomes large and the branch for $T > T_c$ approaches to a slope $-\frac{7}{8}$ line. The best-fit values of the slope of these two branches (from $t L^{\frac{1}{\nu}} = 4$ to 10) for a large size system, $L = 256$, are
0.119 ± 0.002 and −0.816 ± 0.017, which are very close to the predicted values $\frac{1}{8}$ and $-\frac{7}{8}$.

In Figure 2.4, the branches for $T < T_c$ and $T > T_c$ approach to slope $-\frac{7}{4}$ lines as $tL^{\frac{1}{\nu}}$ becomes large. The best-fit values of the slope of these two branches (from $tL^{\frac{1}{\nu}} = 5$ to 16) for a large size system, $L = 256$, are $-1.654 \pm 0.020$ and $-1.892 \pm 0.073$, which are very close to the predicted value $-\frac{7}{4}$. The critical temperature used here is, $T_c = 2.269$, which is the exact value for 2-dimensional square lattice Ising model.

One important question is how to determine the critical temperature $T_c$. A widely used method is the crossing method. The Binder order cumulant, $Q$, or the Binder 4th order cumulant, $U_4$, is used to obtain the critical temperature, $T_c$. The critical temperature is determined by the crossing point of $Q$, or $U_4$, for large size systems. Here $Q$ and $U_4$ are defined as

$$Q \equiv \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2}$$

and

$$U_4 \equiv 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}.$$  \hspace{1cm} (2.152)

The Binder order cumulant $Q$ and the Binder 4th order cumulant $U_4$ are thermodynamic quantities in terms of $T$ and $L$, and they scale as,

$$Q(T, L) = f_{Q, \xi} \left( \frac{\xi(T, L)}{L} \right)$$

$$U_4(T, L) = f_{U, \xi} \left( \frac{\xi(T, L)}{L} \right).$$ \hspace{1cm} (2.154)

If the reduced temperature $t$ is used instead of $T$, then

$$Q(t, L) = f_{Q, \xi, \infty} \left( \frac{\xi_{\infty}}{L} \right) = f_Q(tL^{\frac{1}{\nu}})$$

$$U_4(t, L) = f_{U, \xi, \infty} \left( \frac{\xi_{\infty}}{L} \right) = f_U(tL^{\frac{1}{\nu}}).$$ \hspace{1cm} (2.155)
Figure 2.3 Example of the order parameter $|M|$ scaling for the 2d Ising model with $\beta = \frac{1}{8}$

Note: Square lattice, $|M| = L^{-\frac{\beta}{\nu}} f_M(tL^\frac{1}{\nu})$, $\beta = \frac{1}{8}$, $\nu = 1$. 
Figure 2.4 Example of the susceptibility $\chi$ scaling for the 2d Ising model with $\gamma = \frac{7}{4}$, $\nu = 1$.

Note: Square lattice, $\chi = L^{\gamma} f_x (tL^{\nu})$, $\gamma = \frac{7}{4}$, $\nu = 1$. 
and

\[ U_4(t, L) = f_{U, \xi_\infty} \left( \frac{\xi_\infty}{L} \right) = f_u(t L^{\frac{1}{\nu}}). \] (2.156)

For large size systems, it is expected that

\[ \lim_{L \to \infty} \frac{\xi(T, L)}{L} = \text{constant}, \] (2.157)

so the solution of

\[ Q(T_c, L_1) = Q(T_c, L_2) \] (2.158)

or

\[ U_4(T_c, L_1) = U_4(T_c, L_2) \] (2.159)

gives a good estimate of the critical temperature, \( T_c \). One of the approaches to determine \( \nu \) is using the scaling relation of \( \xi \) near the critical point. Since \( \xi \) scales as

\[ \xi(T, L) \sim L, \] (2.160)

then

\[ \frac{\xi(T_2, L_2)}{\xi(T_1, L_1)} = \frac{L_2}{L_1}, \] (2.161)

and from Equation (2.138)

\[ \xi \sim t^{-\nu} \] (2.162)

\[ \frac{\xi(T_2, L_2)}{\xi(T_1, L_1)} = \frac{t_2^{-\nu}}{t_1^{-\nu}}, \] (2.163)

which leads to

\[ t_2 = \left( \frac{L_1}{L_2} \right)^{\frac{1}{\nu}} t_1. \] (2.164)
2.5 Small world networks

Complex networks [48] describe a wide range of systems in nature and society such as the Internet, social networks, chemical cells, integrated circuits and micro-structures of materials. In recent years, these systems have been modeled no longer as random graphs [8] but rather as systems with robust organizing structures. Their topology and property are governed by some principles. Statistical mechanics methods offer an ideal framework for describing these systems [1].

These developments introduced new concepts and methods to solve challenging problems in statistical physics and condensed-matter physics. The small-world concept describes the fact that there is a relatively short path between any two nodes in a network, even though the network may have a very large size. A small fraction of random long-range small-world connections (interactions) may introduce novel physics, especially in magnetic materials. Immediately following the famous work of Watts and Strogatz [65], many magnetic models based on small-world connections were introduced.

The original small-world model of Watts and Strogatz [65] is built from a $d$-dimensional regular lattice by randomly rewiring each edge of the lattice with a probability of $p$. In a slightly modified small-world model, the random connections are added instead of rewired [51].

To obtain a small world connection, start from a pure lattice with nearest neighbor nodes bonded together. Then between pairs of nodes randomly chosen from the underlying lattice, extra links, which are called shortcuts or small-world connections, are added while
Figure 2.5 Examples of Small World (SW) lattices

Note: In the figure, the filled circles are the locations of the Ising spins, the light solid bonds have strength $J_1$ and the dashed bonds (the SW bonds) have strength $J_2$. (a) A one-dimensional SW graph with $N = 12$ and three randomly chosen SW bonds. (b) A square-lattice with $L = 4$ with $L$ SW bonds.

no links are removed from the underlying lattice. If two nodes are bonded together, they are called nearest neighbors. Figure 2.5 shows examples of small-world networks based on 1-dimensional and 2-dimensional lattices. This method of obtaining small world lattices is different from the original one in that no region of the lattice will be cut and left to not interact with the rest of the lattice (which is usually called clustering) [51].

Consider a $d$-dimensional lattice of linear dimension $L$, thus totally $L^d$ nodes, with periodic boundary conditions. If $w$ shortcuts are added, the average nearest neighbor number $z$ will be

$$
z = 2d + \frac{2w}{L^d} = 2d \left(1 + \frac{w}{dL^d}\right) = 2d(1 + \phi). \quad (2.165)$$
Since the original lattice has $dL^d$ nearest neighbor bonds,

$$\phi = \frac{w}{dL^d}$$

is the ratio of small-world bonds to the original nearest neighbor bonds.

An important quantity in a network is the average length of the distance between any two nodes, $l$, which is defined as the number of inter nodes from one node to another. The relation between $l$ and the lattice size $L$, or $N$, is complicated. A length scale $\xi$ is introduced to characterize small world lattices. There are many ways to define $\xi$ [47]. A convenient definition is [51]

$$\xi = \frac{1}{(\phi d)^{\frac{1}{d}}}.$$  \hspace{1cm} (2.167)

This quantity is similar to the correlation length in the Ising model discussed in the previous sections.

Barthélymé and Amaral [6] first discussed the scaling relation of $l$ with $\xi$ and gave a form $l = \xi G(L/\xi)$. Newman and Watts concluded that for small $\phi$ or $L$, $l$ scales linearly to $L$ while for large $\phi$ or $L$, $l$ scales logarithmically as $l \sim \log N = d \log L$ [51]. Watts pointed out that if $\phi$ is large enough so that the average number of shortcuts is larger than one, $l$ scales logarithmically with $L$; otherwise if $\phi$ is small so that the average number of shortcuts is smaller than one, $l$ scales linearly with $L$. Thus at the crossover point between the logarithmic scaling region and the linear scaling region the number of shortcuts is one [64], so

$$\phi dL^d = 1$$  \hspace{1cm} (2.168)
and

$$L = \frac{1}{(\phi d)^{\frac{1}{2}}} = \xi. \quad (2.169)$$

Thus \( l \) scales with \( L \). Newman and Watts suggested that the relation between \( l \) and \( L \) has the form [47],

$$l = \frac{L}{2dz}F(\phi z L^d), \quad (2.170)$$

where \( F(x) \) has the property,

$$F(x) = \begin{cases} 
1 & x \ll 1 \\
\frac{\log x}{x} & x \gg 1.
\end{cases} \quad (2.171)$$

In the one-dimensional case, Newman and Watts constructed a renormalization group transformation for the small-world model and demonstrated the renormalization results are the same as the above results in the limit of large system size [50]. Newman, Moore and Watts obtained an exact mean-field solution of \( l \) for the one-dimensional small-world model [49].

### 2.6 Monte Carlo for Ising model

Monte Carlo methods are estimation methods used to solve problems, such as numerical integration of a given function, differential equation, flow transportation, radioactivity decay, and stochastic process simulations. In this dissertation Monte Carlo simulation of magnetic systems will be the area of concentration, in particular that of the Ising model. Since the static behavior of the model is studied in this work, static rather than dynamic Monte Carlo simulation is used.
2.6.1 Standard Monte Carlo for the Ising model

The simplest form of Monte Carlo simulation applied to a statistical problem includes three steps. First, generate pseudo-random number(s). Second, depending on the number(s) generated, choose a trial move from the current state to a new state. Third, decide whether to accept or reject this trial move by comparing to an acceptance probability with the random number(s).

In Monte Carlo, certain trial moves are considered to be the effect of physical interactions. Consider the Ising model mentioned in the previous sections, a single spin flip (a trial), i.e. \( s_i \to -s_i \), can be considered to be due to the interaction between the Ising spins or a heat bath of other spins [55] [52].

A useful algorithm for the kinetic Ising model used by Martin [44] in 1977 in a lattice of two-state particles is:

1) Randomly choose an Ising spin with equal probability for all \( N \) spins.

2) Generate a random number \( r \).

3) Calculate (or look up in a stored table) the energy \( E_{old} \) of the current Ising configuration and \( E_{new} \) of the configuration with the chosen Ising spin flipped.

4) Change to the new configuration, i.e. flip the chosen Ising spin, if

\[
    r \leq \frac{\exp^{-E_{new}/k_BT}}{\exp^{-E_{old}/k_BT} + \exp^{-E_{new}/k_BT}}.
\]

(2.172)

5) Take measurements.

6) Increment the time by one Monte Carlo step (mcs).
Here $T$ is the system temperature and $k_B$ is the Boltzmann’s constant. This algorithm, in particular the choice of Equation (2.172), was introduced by Glauber [16] to study the dynamics of a one-dimensional Ising chain and is called the Glauber dynamic. Another popular dynamic algorithm, the Metropolis dynamic [45], is also used for the flip probability for Ising spins and can also obtain the correct statics for the Ising model.

Quantities measured in this work are the magnetization density $M$, the absolute value of the magnetization density (used as the order parameter per spin in finite size Monte Carlo) $|M|$, the susceptibility per spin $\chi$, the specific heat per spin $C_H$, the Binder 4th-order cumulant $U_4$ and the internal energy density $E$. At low temperature, the magnetization obtained from Monte Carlo simulation, $M$, approaches to zero for finite size lattices, Equation (2.23) doesn’t give a correct result of $\chi$. It is convenient to use $|M|$ as the order parameter and treat $\chi$ as a function of $|M|$. Statistics are over $N_{\text{trial}}$ trials for an $N_{\text{tot}}$ spins system. From Equation (2.20), Equation (2.21), Equation (2.22), Equation (2.23), and Equation (2.25), the following equations are obtained. In these equations, $J_{ij}$ is the interaction intensity over the connection between spin $i$ and spin $j$. The interaction intensity is $J_{ij} = J_1$ for regular lattice nearest neighbor site connections and $J_{ij} = J_2$ for small world connections. For the $k$th trial, the spins at sites $i$ and $j$ have values $s_{ik}$ and $s_{jk}$. A quantity symbol $S$ with a subscript $k$, $S_k$, represents the value of the quantity $S$ at the $k$th trial. The summation $\sum_{(i,j)}$ extends over all nearest neighbor bonds.

$$E = \frac{1}{N_{\text{tot}}} \left\langle \sum_{(i,j)} (-J_{ij} s_i s_j) \right\rangle$$
\[ E_k = \frac{1}{N_{\text{tot}}} \sum_{(i,j)} (-J_{ij}s_is_j) \quad (2.174) \]

\[ C_H = \frac{1}{N_{\text{tot}}} \left( \frac{1}{k_B T^2} \left( \frac{1}{N_{\text{trial}}} \sum_k (E_k^2 - \frac{1}{N_{\text{trial}}} \sum_k E_k^2 - \frac{1}{N_{\text{trial}}} \sum_k E_k^2)^2 \right) \right) \]

\[ M = \frac{1}{N_{\text{tot}}} \left\langle \sum_{i=1}^{N_{\text{tot}}} s_i \right\rangle \]

\[ M_k = \frac{1}{N_{\text{tot}}} \sum_{i=1}^{N_{\text{tot}}} s_{ik} \quad (2.177) \]

\[ |M| = \frac{1}{N_{\text{tot}}} \left\langle |\sum_{i=1}^{N_{\text{tot}}} s_i| \right\rangle \]

\[ = \frac{1}{N_{\text{tot}}} \frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} \frac{1}{N_{\text{tot}}} |\sum_{i=1}^{N_{\text{tot}}} s_{ik}| \]

\[ = \frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} \frac{1}{N_{\text{tot}}} |\sum_{i=1}^{N_{\text{tot}}} s_{ik}| \]

\[ = \frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} |M_k| \quad (2.178) \]
\[
U_4 = 1 - \frac{\left\langle \left( \sum_{i=1}^{N_{\text{tot}}} s_i \right)^4 \right\rangle}{3 \left\langle \left( \sum_{i=1}^{N_{\text{tot}}} s_i \right)^2 \right\rangle^2}
\]

\[
= 1 - \frac{\frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} \left( \sum_{i=1}^{N_{\text{tot}}} s_{ik} \right)^4}{3 \left( \frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} \left( \sum_{i=1}^{N_{\text{tot}}} s_{ik} \right)^2 \right)^2}
\]

\[
= 1 - \frac{\frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} \left( M_k N_{\text{tot}} \right)^4}{3 \left( \frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} \left( M_k N_{\text{tot}} \right)^2 \right)^2}
\]

\[
= 1 - \frac{1}{3} \frac{\sum_{k=1}^{N_{\text{trial}}} M_k^4}{\left( \sum_{k=1}^{N_{\text{trial}}} M_k^2 \right)^2}
\]  

(2.179)

\[
\chi_T = \frac{1}{N_{\text{tot}}} \frac{\left\langle \left( \sum_{i=1}^{N_{\text{tot}}} s_i \right)^2 \right\rangle - \left\langle \sum_{i=1}^{N_{\text{tot}}} s_i \right\rangle^2}{k_B T}
\]

\[
= \frac{1}{N_{\text{tot}}} \frac{\frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} \left( M_k N_{\text{tot}} \right)^2 - \left( \frac{1}{N_{\text{trial}}} \sum_{k=1}^{N_{\text{trial}}} M_k N_{\text{tot}} \right)^2}{k_B T}
\]

\[
= \frac{N_{\text{tot}}}{k_B T N_{\text{trial}}} \left( \sum_{k=1}^{N_{\text{trial}}} M_k^2 - \frac{1}{N_{\text{trial}}} \left( \sum_{k=1}^{N_{\text{trial}}} M_k \right)^2 \right)
\]  

(2.180)

\[
\chi_T(|M|) = \frac{N_{\text{tot}}}{k_B T N_{\text{trial}}} \left( \sum_{k=1}^{N_{\text{trial}}} M_k^2 - \frac{1}{N_{\text{trial}}} \left( \sum_{k=1}^{N_{\text{trial}}} |M_k| \right)^2 \right)
\]  

(2.181)

(2.182)

The statistics must be taken over a large number of trials related to the system size. In the work of this dissertation, after a random configuration had been generated, \(10^6\) Monte Carlo steps per spin (MCSS) were taken for system thermalization and statistics were taken over at least an additional \(10^6\) MCSS (up to \(10^9\) MCSS for large system sizes and near the critical point).
2.6.2 Parallelized Monte Carlo, uncorrelated random number generation – SPRNG

For complicated statistical systems, one trial of a change from one state to another state (one Monte Carlo step per spin, defined to be MCSS) will require a large amount of computer time. One can not wait to finish a simulation on a single-CPU computer. Implementing a Monte Carlo algorithm on parallel computers [52] is an efficient way to solve this problem. A parallel computer has hundreds to thousands of processing elements (PEs) and can lead to short execution times. In the simplest parallelization scheme, the configuration of the system being simulated is constructed and initialized on one PE and then broadcast to other PEs. Each PE runs a number of trials independently and no communication between PEs occurs during the simulation. After the simulations complete on all PEs, one PE performs collection and analysis. But if the system size is too large to fit onto a single PE, a more complicated parallelization scheme [52] [55] is required.

Applying parallelized Monte Carlo simulation to the Ising model requires a reliable pseudo random number generation. The generator to be used must provide a practically infinite number of parallel pseudorandom number streams with favorable statistical properties within and among the streams. SPRNG [4] is a scalable parallel pseudo random number generating package which provides generators satisfying the above requirements. With a new standard library for scalable pseudorandom number generation, SPRNG provides users with various SPRNG random number generators. It can be easily used on a variety of architectures, especially in large-scale parallel Monte Carlo applications. The SPRNG package has the following advantages:
1) The pseudorandom number streams generated by SPRNG can be absolutely reproduced for computational verification. It does not depend on the number of processors used in the computation.

2) Unique pseudorandom number streams are dynamically created on each PE without interprocessor communication.

3) Several types of pseudorandom numbers are available, all with good statistical properties.

The simplest parallelization scheme (with up to 128 PEs) along with SPRNG pseudorandom number generator 2.0 is used in the work presented in this dissertation. In particular, each PE performs its own independent simulation for its value of temperature, and the results for the quantities like the magnetization density $M(T)$, the internal energy density $E(T)$, the Binder 4th-order cumulant $U_4(T)$, the susceptibility per spin $\chi(T)$, and the specific heat per spin $C(T)$ are written to a file once the computation finishes. The program is written in C, and the calculations are according to Equation (2.174), Equation (2.173), Equation (2.175), Equation (2.177), Equation (2.176), Equation (2.178), Equation (2.179), Equation (2.180), and Equation (2.181).
CHAPTER III

THE $z$ NEAREST-CONNECTION ISING MODEL

The interaction range and the dimensionality determine the critical behavior of a system. Magnetic systems with small-world connections can be considered to be a flat system with small-world short-cuts added. It is still not clear how the short-cuts (small-world connections) affect the critical properties of these systems, such as the scaling properties of the magnetization $M$, the susceptibility $\chi$, and the specific heat $C$.

For better understanding small-world magnetic systems, it is useful to construct an Ising model with only random short-cuts, or long-range connections, thus the properties of such a system are only affected by the small-world short-cut interactions. How such a system behaves is important to the studies of a small-world magnetic system. In this chapter, a small-world model, which will be called the $z$ nearest connection Ising model, or the $z$-model, will be constructed and its critical behavior will be studied. This model is very similar to a model that was first introduced in ‘Critical Properties of an Ising Model with Dilute Long Range Interactions’ by R. Scalettar [62] as will be shown here. But the scaling forms he introduced do not give convincing scaling. A form of scaling that does work will be discussed.
3.1 The $z$ nearest-connection Ising model construction

The model introduced here is an Ising lattice with each spin having $z$ randomly placed connections to other spins. It is based on a two dimensional square lattice (for simplicity, the lattice size, $L$, is even) with each site occupied by a $s = \pm 1$ Ising spin. At the starting point, there is no interaction between any spins (this would correspond to $L^2$ non-interacting spins). Then short-cuts are added to these spins. Below is the algorithm to generate an Ising model in which each spin having $z$ short-cuts and all $L^2$ spins belonging to the same interacting cluster. The phrase ‘random’ means with equal probability.

1. Construct a $z = 2$ network.

   - Generate a $2d$ square lattice with no connections among the spins, let $N$, the total number of spins be even.
   - Begin with a randomly chosen Ising spin, randomly choose another spin, add a short-cut between them.
   - Randomly choose a spin from the remaining spins that have not been chosen, add a short-cut between it and the spin chosen in the previous step.
   - Repeat the above operation until all spins are connected to other spins, then add a short-cut between the last spin to be chosen and the starting spin. So far, a $z = 2$ model is constructed. It is equivalent to a 1-dimensional ring. Following this procedure, the system constructed definitely has only one interacting cluster.
2. Randomly add one more short-cut to each spin according to the following procedure for \( z > 2 \).

- Randomly choose 2 different spins and add a short-cut between them.
- From the remaining spins that have not been chosen, randomly choose 2 different spins and add a short-cut between them.
- Repeat the above operation until all the spins have one more short-cut.

3. Repeat the above step until every spin has \( z \) interactions.

Figure 3.1 shows an equivalent 1d-ring for \( z = 2 \), a 1d-ring with 1 small-world connections added, a 1d-ring with 2 small-world connections added, and a 1d-ring in which each spin has a small-world connections (\( z = 3 \)). In the model constructed here, two spins may be connected directly through one or more short-cut(s). If two spins are connected and interact directly through at least one short-cut, they are called nearest neighbor spins (nn spins) to each other. Thus most of the spins in the model have \( z \) nearest neighbor spins, a few of them has less than \( z \) nearest neighbor spins because of possible multiple connections. But all the spins in the model interact with their nearest neighbor spins through \( z \) short-cuts. So this model is called the \( z \) nearest-connection Ising model, or the \( z \)-model.

The interactions via the short-cuts are assumed to be the same, equal to \( J \), and the interactions between non-nearest neighbors are assumed to be zero. Thus, in zero field the Hamiltonian of the system is,

\[
\mathcal{H} = -J \sum_{(ij)} s_is_j, \tag{3.1}
\]
Figure 3.1 Examples of the $z$-model lattices

Note: The lattices shown are the equivalent $1d$-ring for $z = 2$ (a), $1d$-ring with 1 small-world connection (b), $1d$-ring with 2 small-world connections (c), and $1d$-ring with $\frac{N}{2}$ small-world connections ($z = 3$) (d).
where $s_i = \pm 1$, and the summation extends over all short-cuts between nearest neighbor pairs $\langle ij \rangle$. If there are $k$ short-cuts between a pair of nearest neighbor spins, then the interaction between these two spins is equivalent to one interaction of strength $kJ$.

### 3.2 Monte Carlo simulation results

The Monte Carlo simulation for the $z$ model is performed for $z = 3, 4, 5, 6, 7, 8$ nearest connections with lengths of the square lattice

$$L = 16, 32, 64, 128, 256, \quad (3.2)$$

so the total spins are

$$N = L^2 = 256, 1024, 4096, 16384, 65536. \quad (3.3)$$

Static Monte Carlo is performed and the Glauber algorithm is used. This work is done on a parallel computer, Empire. Empire is an IBM Linux Cluster at ERC, Mississippi State University. The Empire cluster is a 1038-processor supercluster consisting of 519 IBM X330 nodes, each with two Intel Pentium III 1.266 GHz processors and 607 gigabytes (GB) of RAM. The nodes are interconnected via 16 Extreme Networks Summit 48si 48-port switches and one Extreme Networks Summit 7i 32-port Gigabit switch. Empire has a peak performance of 1.3 teraflops. By using 64 PEs (up to 128 PEs for large size and large $z$) each time, a random configuration of lattices with $L^2$ spins and $z$ nearest connections is simulated. Each PE performs its own independent simulation for its value of temperature, but all have the same random arrangement of the connections. In the vicinity of the critical
temperature, the quantities at 64 temperature points are calculated. It takes about 52 hours for the longest simulation, which is a simulation of a \( L = 256, N = 65536 \) lattice in which each spin has \( z = 8 \) nearest connections.

The quantities measured are the order parameter \(|M|\), the Binder 4th order cumulant \( U_4 \), the isothermal susceptibility \( \chi \), \( \chi T \), and the isomagnetic field specific heat \( C_H \). Since \( \chi \) diverges at low temperature, \( \chi T \) is used for later scaling. This can be seen from Figure 3.2, in which \( \chi \) vs. \( T \) and \( \chi T \) vs. \( T \) of a 2-dimensional square lattice with \( L = 30 \) are shown. Here the values of \( \chi \) are calculated according to Equation (2.180) which means \( \chi \) is treated as a function of \( M \). In finite size Monte Carlo simulation, if \( 10^6 \) MCSS is taken, \( M \sim 0 \) at low temperature. But in real physical systems, \( M \) is not close to zero at low temperature, \( M \sim |M| \). So if \( M \) is used as the order parameter, one can not obtain correct results. Thus the absolute value of the magnetization, \(|M|\), is used as the order parameter. Then the susceptibility \( \chi \) is treated as a function of \(|M|\), so Equation (2.181) is used to calculate \( \chi \) instead of Equation (2.180). The free energy density \( E \), the order parameter per spin \(|M|\), the Binder 4th order cumulant \( U_4 \), the specific heat per spin \( C_H \) are calculated according to Equation (2.174), Equation (2.173), Equation (2.178), Equation (2.177), Equation (2.176), Equation (2.179), and Equation (2.175).

The results for \( z = 3, 4, 8 \) are plotted in Figure 3.3 to Figure 3.14. Figure 3.15 and Figure 3.16 are the Binder 4th order cumulant \( U_4 \) and the susceptibility times temperature \( \chi T \) for all \( z, z = 3, 4, 5, 6, 7, 8 \). It can be seen from these figures, the peaks of \( \chi T \) and \( C_H \) fall at the temperature corresponding the crossing point of \( U_4 \) for large size.
Figure 3.2 The divergence of $\chi$ and $\chi T$ at low temperatures

Note: Here $\chi$ is a function of $M$ and its values are calculated from Equation (2.180).
systems for each $z$. The critical point $T_c$ obtained from the crossing point of $U_4$ shifts to higher temperature when $z$ is increased. The $T_c$ values are shown in Table 3.1. Here all values are accurate within ±0.005. For each $z$, the peak values of the susceptibility $\chi$ and the specific heat $C_H$ shift to higher temperature as the total number of spins in the lattice $N$ becomes large. The locations of these maxima are near that of the crossing point of the Binder 4th order cumulant $U_4$.

Table 3.1  

<table>
<thead>
<tr>
<th>$z$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_c$</td>
<td>1.821</td>
<td>2.885</td>
<td>3.914</td>
<td>4.933</td>
<td>5.942</td>
<td>6.950</td>
</tr>
</tbody>
</table>

The $M$ vs. $T/T_c$ plot for a large size lattice ($L = 256$) is shown in Figure 3.17. For finite size systems, the magnetization at zero field ($H = 0$) and zero temperature ($T = 0$) is

$$M_0 = M(0,0) = N. \quad (3.4)$$

Substitute $M_0 = N$ into Equation (2.52), and use $M$ to represent the order parameter per spin, the $M$ and $T$ relation can be expressed as

$$M = \tanh \left( M \frac{T_c}{T} \right). \quad (3.5)$$

This equation is used to obtain the theoretical curve in Figure 3.17. In the vicinity below the critical point $T/T_c = 1$, the $M$ vs. $T/T_c$ curves for $z = 3$, $z = 4$, and $z = 8$
Figure 3.3  The simulation result for $U_4$ for the $z$-model for $z = 3$

Note:  The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 3.4 The simulation result for $|M|$ for the $z$-model for $z = 3$
Figure 3.5 The simulation result for $C_H$ for the $z$-model for $z = 3$
Figure 3.6 The simulation result for $\chi T$ for the $z$-model for $z = 3$
Figure 3.7  The simulation result for the Binder 4th cumulant $U_4$ for the $z$-model for $z = 4$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 3.8 The simulation result for the order parameter $|M|$ for the $z$-model for $z = 4$
Figure 3.9 The simulation result for the specific heat $C_H$ for the $z$-model for $z = 4$.
Figure 3.10 The simulation result for the susceptibility $\chi T$ for the $z$-model for $z = 4$
Figure 3.11  The simulation result for the Binder 4th order cumulant $U_4$ for the $z$-model for $z = 8$

Note:  The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 3.12  The simulation result for the order parameter $|M|$ for the $z$-model for $z = 8$
Figure 3.13  The simulation result for the specific heat $C_H$ for the $z$ model for $z = 8$
Figure 3.14 The simulation result for the susceptibility $\chi_T$ for the $z$-model for $z = 8$. 
Figure 3.15  The simulation result for the Binder 4th order cumulant $U_4$ for the $z$-model for $z = 3, 4, 5, 6, 7, 8$

Note:  The exact value at criticality $[9]$ of $U_4 = 0.2705201$ is given by the horizontal line. The critical temperature $T_c$ shifts to a higher value as $z$ is increased.
Figure 3.16 The simulation result for the susceptibility $\chi T$ for the $z$-model for $z = 3, 4, 5, 6, 7, 8$

Note: The peak of $\chi T$ shifts to higher temperature as $z$ is increased and fall in the vicinity region of $T_c$, which can be determined from $U_4$ plot.
are very close to the mean field solution. That suggests the $z$-model may have mean field properties.

### 3.3 Analysis and scaling

The critical behavior of a system is affected by long-range interactions. Scalettar studied the properties of long-range interactions with his Ising model. The model he introduced is similar to the $z$-model in that they both have $z$ randomly chosen nearest neighbor connections. In his model, every spin has $z$ randomly chosen nearest neighbor spins, while in the $z$ model, a few spins may have less than $z$ nearest neighbor spins. Here is how his model is constructed [62].

Suppose we have a total of $V$ sites each of which is to have $z$ randomly chosen neighbors. Our operational prescription for the construction of random neighbors is to begin with a list of integers of length $zV$ in which the first $V$ positive integers are repeated $z$ times. This list is randomized by a large number of pair interchanges. Sites are defined to be neighbors if they appear as the $2n - 1$ and $2n$ entries in the list, $n = 1, 2, \cdots zV/2$. We eliminate by further randomization any pair that appears twice or sites that are self-connected. The end result of this procedure is a lattice in which each spin has $z$ neighbors randomly chosen from the $V - 1$ remaining spins.

So his model is different from the $z$-model in that there is no more than one short-cut between any two of the spins in his model. Since the number of multiple connections in
Figure 3.17 The $z$-model simulation result and mean field prediction $M$ vs. $T/T_c$ comparison

- $z=3$, $T_c=1.821$
- $z=4$, $T_c=2.885$
- $z=8$, $T_c=6.950$
- Mean field
the \( z \) model is very small, it will not affect the system properties significantly. Thus the long-range interactions affect the critical behavior in the same way in the \( z \) model and Scalettar’s model. These two models have similar critical properties and their physical quantities should scale the same.

For long-range random connection models, the average separation between two spins, \( l \), is,

\[
l = a(z) + b(z) \ln N,
\]

where \( N \) is the total number of spins in the lattice. Scalettar discussed the average separation of the Bethe lattice [62] and concluded \( l \) grows with \( N \) logarithmically similar to Equation (3.6) with

\[
b(z) = \frac{1}{\ln(z - 1)}.
\]

He applied this to the high temperature expansion of \( \chi \) and obtained the critical point equation

\[
\tanh(\beta_c J) \approx e^{-\frac{1}{\beta_c(z)}}.
\]

Thus

\[
T_c = \frac{2J}{k_B \ln \frac{z}{z - 2}}.
\]

From Equation (2.47), the mean field result is \( T_c = \frac{\lambda M_0}{k_B} \). Using the mean field approximation to the \( z \)-model, gives \( \lambda \approx \frac{Jz}{N} \). From Equation (2.42), \( M_0 = N \), so \( T_c \approx \frac{Jz}{k_B} \). For \( k_B = 1 \), then \( T_c \approx Jz \). This is the same as Scalettar’s result [62], \( T_c = Jz \). Figure 3.18 shows \( T_c/z \) vs. \( z \) from the Monte Carlo results of several models as well as the prediction
for the Bethe lattice. The $T_c/z$ vs. $z$ points of the $z$-model fall closely to the Bethe lattice curve. The $T_c$ values used here are the crossing point of the Binder 4th order cumulant for large lattice sizes, as described in Equation (2.159).

For long-range interaction models, the average separation length $l$ grows logarithmically with $N$. Scalettar scaled the susceptibility using $l \sim \ln N$ for $z = 3$ in his model. He used $\nu = 0.53$, $\gamma = 1.06$, and $T_c = 1.87$. Figure 3.19 is his result [62].

When Scalettar’s values of $\nu$, $\gamma$ and $T_c$ is applied to the $z$-model for $z = 3$, the scaling result is not good, see Figure 3.20.

Even the $T_c$ is changed from the value given by the $U_4$ crossing from various system sizes, which is $T_c = 1.821$, and the exponents $a$, $b$ are changed step by step to fit the scaling, the scaling result is not good, either. The scaling result for the best $a$ and $b$ is drawn in Figure 3.21. From these results, it can be concluded that although the average separation $l$ grows logarithmically with the number of total spins $N$, the scaling function is not logarithmic with $N$.

From Equation (2.156), the Binder 4th order cumulant scales as

$$U_4(t) = f_u(tL^\frac{\nu}{2}).$$

Using the above scaling equation for the mean field case, following Brézin and Zinn-Justin’s scaling prediction for Ising models (above the upper dimension, $d \geq 4$) [9], Laradji and Landau [34] obtained the Binder 4th order cumulant scaling relation for a system governed by the mean field theory,

$$U_4(t) = f_u(tL^\frac{\nu}{2}),$$
Figure 3.18 Plot of $T_c/z$ vs. $z$ for the $z$-model and 2$d$ small world models.
Figure 3.19  Scalettar’s $\chi$ scaling for the $z$-model

Note: $z = 3$, $T_c = 1.87$, $a = 2.0$, $b = 1.8868$. This is a scan of Fig. 5 from [62].
Figure 3.20  The susceptibility \( \chi \) scaling for the \( z \)-model using Scalettar’s values

Note: For \( z = 3 \), \( T_c = 1.87 \), \( a = 2.0 \), \( b = 1.8868 \) [62].
Figure 3.21  The $\chi T$ scaling using Scalettar’s scaling function for the $z$-model (best fitting)

Note:  For $z = 3$, $T_c = 1.821$, $a = 4.1$, $b = 3.5$. The values of $a$ and $b$ are determined by eye to give the best data collapse scaling.
where \( L^d = N \) is the number of total spins in a \( d \)-dimensional lattice. This scaling was applied to the \( z \)-model data to see whether it scales in a mean field like fashion. Since the \( z \)-model does not have a characteristic length, \( N = L^d \) is used instead of \( L \). Thus the above relation becomes,

\[
U_4(t) = f_u(t N^{1/2}).
\]  

(3.12)

Taking the derivation of the above equation, gives,

\[
\frac{\partial U_4(t)}{\partial t} = N^{1/2} \frac{\partial f_u(t N^{1/2})}{\partial (t N^{1/2})} = N^{1/2} f_u'(t N^{1/2}).
\]  

(3.13)

If the \( z \)-model has mean field properties, it is expected that the value of \( \frac{\partial U_4(t)}{\partial t} \) at \( U_4 = 0.2705201 \), which is the theoretical value for infinite size mean field models [9], is proportional to \( N^{1/2} \). Log-log plots for \( -\frac{\partial U_4(t)}{\partial t} \) vs. \( N \) are given in Figure 3.22 to Figure 3.24. The curves match the expected slope \( \frac{1}{2} \) line very well. The following linear-fitting is performed.

\[
-\frac{\partial U_4(t)}{\partial t} \bigg|_{u_\infty} = a_0 N^{0.5}.
\]  

(3.14)

The best-fit prefactors for various \( z \) are shown in Table 3.2. Figure 3.24 is a plot for \( -\frac{\partial U_4(t)}{\partial t} \) vs. \( N \) for \( z = 3, 4, 5, 6, 7, 8 \).

| Table 3.2 | The best-fit prefactors of \( \frac{\partial U_4(t)}{\partial t} \bigg|_{u_\infty} = a_0 N^{0.5} \) for the \( z \)-model for various \( z \) |
|-----------|----------------------------------|
| \( z \)   | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) |
| \( a_0 \) | 0.096 | 0.058 | 0.042 | 0.040 | 0.029 | 0.021 |
Figure 3.22  The log-log plot of \( \frac{\partial U_4(t)}{\partial t} \bigg|_{u_\infty} \) vs. \( N \) for the \( z \)-model for \( z = 3 \)

Note: Here \( u_\infty = 0.2705201 \), is the theoretical value for infinite size mean field models [9]. The line is the best fit with the predicted slope of \( \frac{1}{2} \).
Figure 3.23 The log-log plot of $\frac{\partial U_4(t)}{\partial t}|_{u_\infty}$ vs. $N$ for the $z$-model for $z = 8$

Note: Here $u_\infty = 0.2705201$, is the theoretical value for infinite size mean field models [9]. The line is the best fit with the predicted slope of $\frac{1}{2}$. 
Figure 3.24 The log-log plot of $\frac{\partial U_4(t)}{\partial t}|_{u_\infty}$ vs. $N$ for the $z$-model for $z = 3, 4, 5, 6, 7, 8$.

Note: Here $u_\infty = 0.2705201$, is the theoretical value for infinite size mean field models [9]. The line has a slope of $\frac{1}{2}$, which is the expected value.
Using Equation (3.12), the Binder 4th order cumulant scaling results for the $z$-model are very good in the neighborhood of the critical temperature obtained from the crossing point of different system sizes. The larger the value of $z$ is, the better the scaling result is. The results for $z = 5$ and $z = 8$ are shown in Figure 3.25 and Figure 3.26. For a large system size, $L = 256$, and various numbers of long-range connections, scaling curves collapse together, which is shown in Figure 3.27.

Since $\alpha = 0$, there is no scaling for $C_H$ without logarithmic corrections [9] [34]. For the order parameter $|M|$ and the susceptibility $\chi$, mean field scaling relations,

$$|M| = N^{-\frac{1}{4}} f_M(t N^{\frac{1}{2}}),$$  \hspace{1cm} (3.15)

and

$$\chi_{|M|} = N^{\frac{1}{2}} f_{\chi}(t N^{\frac{1}{2}})$$ \hspace{1cm} (3.16)

are used. Scaling results are very good, which are shown in Figure 3.28, Figure 3.29, Figure 3.30, and Figure 3.31.

The scaling relations of the order parameter and the susceptibility, Equation (2.148) and Equation (2.149) can be rewritten as

$$M(t, L) = W^{-\beta} f_M(t W),$$  \hspace{1cm} (3.17)

and

$$\chi(t, L) = W^{\gamma} f_\chi(t W).$$  \hspace{1cm} (3.18)

Compare Equation (3.15) to Equation (3.17) to obtain

$$\beta = \frac{1}{2}.$$  \hspace{1cm} (3.19)
Figure 3.25 The Binder 4th order cumulant $U_4$ scaling for the $z$-model. For $z = 3$, $U_4 = f_u(tN^{1/2})$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line. The critical temperature $T_c$ shifts to a higher value as $z$ is increased.
**Figure 3.26** The Binder 4th order cumulant $U_4$ scaling for the $z$-model. For $z = 8$, $U_4 = f_a(tN^{1/2})$

**Note:** The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line. The critical temperature $T_c$ shifts to a higher value as $z$ is increased.
Figure 3.27  The Binder 4th order cumulant $U_4$ scaling for the $z$-model for a large system size $L = 256$ and various long-range connections, $z = 3, 4, 5, 6, 7, 8, U_4 = f_u(tN^{1/2})$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line. The critical temperature $T_c$ shifts to a higher value as $z$ is increased.
Figure 3.28  The order parameter $|M|$ scaling for the $z$-model for $z = 3$

Note:  Here $|M| = N^{-\frac{1}{2}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. 
Figure 3.29  The order parameter $|M|$ scaling for the $z$-model for $z = 8$

Note: Here $|M| = N^{-\frac{1}{2}} f_M(t N^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. 
Figure 3.30  The susceptibility $\chi$ scaling for the $z$-model for $z = 3$

Note: Here $\chi_{|M|} = N^{\frac{z}{2}} f_{\chi}(tN^{\frac{1}{2}})$, thus $\gamma = 1$. 
Figure 3.31  The susceptibility $\chi$ scaling for the $z$-model for $z = 8$

Note: Here $\chi_{|M|} = N^{\frac{1}{z}} f_{\chi}(t N^{\frac{1}{z}})$, thus $\gamma = 1.$
Compare Equation (3.16) to Equation (3.18) to obtain

\[ \gamma = 1. \]  

(3.20)

It can be seen, the exponents \( \beta \) and \( \gamma \) have the mean field critical exponent values. This means the \( z \)-model scales in a mean field like fashion. Also for large \( tN^{\frac{1}{2}} \) with \( N \gg 1 \) and \( t \ll 1 \), from Equation (3.15) and Equation (3.16), and use Equation (2.29), Equation (2.30), Equation (2.147), it can be obtained that

\[ f_M(tN^{\frac{1}{2}}) \sim (tN^{\frac{1}{2}})^{\frac{1}{2}} \]  

(3.21)

and

\[ f_\chi(tN^{\frac{1}{2}}) \sim (tN^{\frac{1}{2}})^{-1}. \]  

(3.22)

Log-log plots of \( |M| \) scaling, Equation (3.15), for \( z = 3 \) and \( z = 8 \) are drawn in Figure 3.32 and Figure 3.33. At large \( tN^{\frac{1}{2}} \), the below \( T_c \) branches of \( |M| \) scaling approach to a slope \( \frac{1}{2} \) line, and the above \( T_c \) branches approach to a slope \( -\frac{1}{2} \) line. The best-fit values of the slope of these two branches (from \( tN^{\frac{1}{2}} = 4 \) to 10) for a large size system, \( L = 256 \), are \( 0.472 \pm 0.002 \) and \( -0.477 \pm 0.009 \) for \( z = 3 \) (Figure 3.32), \( 0.489 \pm 0.001 \) and \( -0.468 \pm 0.009 \) for \( z = 8 \) (Figure 3.33), which are very close to the expected values \( \frac{1}{2} \) and \( -\frac{1}{2} \). Log-log plots of \( \chi T \) scaling, Equation (3.16), are drawn in Figure 3.35 and Figure 3.36. At large \( tN^{\frac{1}{2}} \), the below \( T_c \) branches of \( \chi T \) scaling approach to a slope \( -1 \) line. The best-fit values of the slope of this branch (from \( tN^{\frac{1}{2}} = 4 \) to 12) for a large size system, \( L = 256 \), are \( -0.954 \pm 0.019 \) for \( z = 3 \) (Figure 3.35) and \( -0.942 \pm 0.024 \) for \( z = 8 \) (Figure 3.36), which are close to the expected value \( -1 \). Figure 3.34 and Figure 3.37
show the scaling for 2 large system sizes and various numbers of long-range connections, the curves for different $z$ fall in a nearby region, that means the scaling relation work well.

From the scaling of $U_4, |M|$ and $\chi$ for the $z$ nearest-connection model, $\beta = \frac{1}{2}$ and $\gamma = 1$ were obtained. These are the same values of the critical exponents of the mean field theory. From the above procedure, for the $z$-model, the order parameter $|M|$ and the susceptibility $\chi$ scale in the same fashion predicted by the mean field theory near the critical point $T_c$. Thus it can be concluded that the $z$ nearest-connection model has mean field properties in the vicinity of the critical point $T_c$. 
Figure 3.32  The order parameter $|M|$ scaling for the $z$-model for $z = 3$, log-log

Note:  Here the scaling relation is $|M| = N^{-\frac{1}{2}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Note: Here the scaling relation is $|M| = N^{-\frac{1}{2}} f_M(t N^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 

Figure 3.33 The order parameter $|M|$ scaling for the $z$-model for $z = 8$, log-log
Figure 3.34 The order parameter $|M|$ scaling for the $z$-model for various system sizes, $L = 128, 256$, and various long-range connections, $z = 3, 4, 6, 8$, log-log.

Note: Here the scaling relation is $|M| = N^{-\frac{4}{z}} f_M(tN^{\frac{1}{z}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 3.35  The susceptibility $\chi$ scaling for the $z$-model for $z = 3$, log-log

Note: Here the scaling relation is $\chi_{|M|} = N^{\frac{z}{2}} f_\chi(t N^{\frac{1}{2}})$, thus $\gamma = 1$. The straight line has a slope of $-1$. 
Figure 3.36 The susceptibility $\chi$ scaling for the $z$-model for $z = 8$, log-log

Note: Here the scaling relation is $\chi_{|M|} = N^{\frac{1}{2}} f_\chi(tN^{1/2})$, thus $\gamma = 1$. The straight line has a slope of $-1$. 
Figure 3.37  The susceptibility $\chi$ scaling for the $z$-model for various system sizes, $L = 128, 256$, and various long-range connections, $z = 3, 4, 6, 8$, log-log

Note: Here the scaling relation is $\chi_{|M|} = N^{\frac{1}{2}} f_{\chi}(tN^{\frac{1}{2}})$, thus $\gamma = 1$. The straight line has a slope of $-1$. 

\[
\chi T/N^{1/2} = N^{-1/2} f_{\chi}(tN^{1/2})
\]

\[
\chi T/N^{1/2} = N^{-1/2} f_{\chi}(tN^{1/2})
\]
CHAPTER IV
THE SMALL WORLD ISING MODEL

The scaling results in the previous chapter show that the \( z \)-model has mean field properties in the neighborhood of the critical temperature. The \( z \)-model is a system with all connections having a long-range character. Small world networks have both nearest-neighbor and long-range (or short-cuts) interactions as connections, and even a small amount of short-cuts may change the properties of a system. But how these small-world connections affect the properties of small-world Ising systems is still a problem to be better solved. In this chapter, small-world Ising models with different amounts of long-range interactions will be constructed and their critical properties will be studied.

4.1 The small world Ising models and their construction

The small-world Ising models (SW models) studied in this dissertation are Ising models based on a two dimensional square lattice (for simplicity, the lattice size, \( L \), is even) with each site occupied by a \( s = \pm 1 \) Ising spin. At the starting point, there is an interaction connection between the spins located on the nearest neighbor sites, no other connection exist. Then short-cuts are added to these spins randomly with equal probability. For a
better understanding of the small-world effects on the critical properties, two kinds of small-world Ising models are constructed in this thesis.

In the first kind of small-world Ising model, there are $w$ small-world connections (long-range short-cuts) added to an $L \times L$ lattice, so each spin has $z = 4 + \frac{2w}{L^2}$ nearest neighbor connections, hence $z \geq 4$. If $z = 4$, there are no small-world connections, the model degenerates to the regular $2d$ square lattice Ising model. For $w \leq \frac{L^2}{2}$, some spins have 1 small-world connection and other spins do not have any small-world connections. For $w \geq \frac{L^2}{2}$, each spin has 1 or more small-world connections. When $z \geq 5$ and $z$ is an integer, this kind of model can be seen as a modified $z$-model. Below is the algorithm to generate a first kind of small-world Ising model.

1. Construct a 2-dimensional network based on a regular $2d$ square lattice with connections existing between the spins on the nearest neighbor sites. So far, each spin has $z = 4$ nearest neighbor connections.

2. Randomly add short-cuts to the lattice according to the following procedure.

   - Randomly choose 2 different spins and add a short-cut between them.

   - From the remaining spins that have not yet had a short-cut connection added in this round, randomly choose 2 different spins and add a short-cut between them.

   - Repeat the above operation until all the spins have one more short-cut or totally $w$ small-world connections are added.
3. Repeat the above step until \( w \) small-world connections are added.

The small-world properties of a system may be affected by the small-world interaction strength. In this model, regular square lattice interactions are assumed to be the same, equal to \( J_1 \), small-world interactions are assumed to be \( J_2 \), and non-nearest neighbor interactions to be zero. Thus, in zero field the Hamiltonian of the system is,

\[
\mathcal{H} = -J_1 \sum_{nn(ij)} s_is_j - J_2 \sum_{SW(ij)} s_is_j, \tag{4.1}
\]

where \( s_i = \pm 1 \). The first summation extends over all spin pairs on nearest neighbor sites and the second summation extends over all small-world bonds.

In the second kind of small-world Ising model, the small-world connections are constructed from linear strands of Ising spins rather than single bonds. These are called physical small world connections. For each small-world bond constructed as in the first model between spins \( i \) and \( j \), \( r_{ij} + 1 \) Ising spins (called small-world Ising spins) are added and the neighbors are coupled with interaction strength \( J_3 \). The spins at the ends of the additional Ising chains are coupled to the corresponding Ising spin on the original lattice site with interaction strength \( J_3 \). Here \( r_{ij} \) is the length of the small-world bond between spins \( i \) and \( j \) rounded to the upper integer. Since periodic boundary conditions are used, the largest small-world bond length is no more than the upper integer of \( \sqrt{2}L \). Thus, in zero field, the Hamiltonian is,

\[
\mathcal{H} = -J_1 \sum_{nn(ij)} s_is_j - J_3 \sum_{SW(ij)} s_is_j. \tag{4.2}
\]
Here \( s_i = \pm 1 \), \( J_1 \) is the regular lattice interaction strength and \( J_3 \) is the interaction strength between the nearest neighbor small-world spins. The interactions between other spin pairs are zero. The first summation extends over all spin pairs of nearest neighbor sites and the second summation extends over all spins on the small-world connections.

### 4.2 Monte Carlo simulation and analysis of small world models

The number of small-world connections affects the critical properties of a magnetic system as well as does the small-world interaction strength. To study these affects, the Monte Carlo simulation for small-world models in this dissertation is performed for different numbers of small-world connections and interaction strengths with length series

\[
L = 16, \ 32, \ 64, \ 128, \ 256, \ 384, \quad (4.3)
\]

so the total numbers of regular-lattice spins are

\[
N = L^2 = 256, \ 1024, \ 4096, \ 16384, \ 65536, \ 147456. \quad (4.4)
\]

Also, similar to the simulations of the \( z \) model, static Monte Carlo is performed and the Glauber algorithm is used. These studies are performed on the same computer as used for the \( z \)-model simulations. By using 64 PEs (up to 128 PEs for large system sizes, large numbers of small-world connections and small-world spins) each time, a random configuration of lattices with \( L^2 \) normal-lattice spins and a different number of small-world connections is simulated.
Each PE performs its own independent simulation for its value of temperature. In the vicinity of the critical temperature, the quantities at 64 temperature points are calculated for small size systems and 120 temperature points for large size systems. It takes about 55 hours for a simulation of a $L = 256$, $N = 65536$ system in which each spin has 4 small-world connections and 63 hours for a simulation of a $L = 256$, $N = 65536$ system with $L$ small-world spin-bonds.

The quantities measured are the order parameter $|M|$, the Binder 4th order cumulant $U_4$, the isothermal susceptibility $\chi$, $\chi^T$, and the isomagnetic field specific heat $C_H$. Here, the susceptibility $\chi$ is a function of $|M|$, and Equation (2.181) is used. The free energy density $E$, the order parameter per spin $|M|$, the Binder 4th order cumulant $U_4$, and the specific heat per spin $C_H$ are calculated according to Equation (2.174), Equation (2.173), Equation (2.178), Equation (2.177), Equation (2.176), Equation (2.179), and Equation (2.175).

4.3 Simulation results and analysis of the first kind of small-world model

Systems with different density of small world connections may behave differently. As the system size becomes large, the density of long-range connections may approach to zero or a constant. Also, systems with different long-range interaction strength may have different critical behavior. In this section, the first kind of small-world model with both vanishing and non-vanishing density long-range connections is studied for strong long-
Figure 4.1  The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $z = 5$

Note:  The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.2 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $z = 5$
Figure 4.3 The simulation result for the susceptibility $\chi^T$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $z = 5$
Figure 4.4 The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $z = 5$. 

$J_1 = 1$, $J_2 = 1$, $z=5$
Figure 4.5  The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $z = 6$

Note:  The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.6 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1, J_2 = 1$) for $z = 6$.
Figure 4.7  The simulation result for the susceptibility $\chi T$ for the first kind of SW model ($J_1 = 1, J_2 = 1$) for $z = 6$. 
Figure 4.8 The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $z = 6$.
Figure 4.9 The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1, J_2 = 1$) for $z = 7$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.10 The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $z = 8$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.11  The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $z = 5$

Note:  The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.12 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $z = 5$
Figure 4.13 The simulation result for the susceptibility $\chi T$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $z = 5$
Figure 4.14 The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $z = 5$. 
range interactions. As for weak long-range interactions, systems with a non-vanishing density of small world connections are studied.

### 4.3.1 Non-vanishing density strong small world connections $z \geq 5$, and $w \propto L^2$

In the $z$-model, the interaction strength for all nearest neighbors are the same. If the interaction strength via small-world bonds has the same value as that via the bonds connected from and to the nearest neighbors, and all spins have same number of nearest neighbor connections, small-world lattices are comparable to the $z$-model. In this section, in order to study the strong small-world interaction effects and compare to the $z$-model, small-world Ising systems are constructed such that all spins have the same number of nearest neighbor connections, thus the long-range connection density is non-vanishing. The small-world interaction strengths are set to be the same as or larger than the regular lattice interaction.

Simulations for the first kind of small-world model are done for systems in which each spin has 1, 2, 3, or 4 small-world connections, or say each spin has $z = 5, 6, 7, 8$ nearest neighbor connections. From Equation (2.166), the long-range connection densities are $\phi = \frac{w}{dL^2} = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. Interaction strengths in the Hamiltonian, Equation (4.1), are set to be $J_1 = 1$, $J_2 = 1$ and $J_1 = 1$, $J_2 = 4$. Thus compare to the regular lattice interactions, the small-world interaction is strong. Figure 4.1 to Figure 4.7 are the simulation results of the order parameter $|M|$, the Binder 4th order cumulant $U_4$, the isothermal susceptibility times the temperature $\chi T$, and the isomagnetic field specific heat $C_H$ for
Figure 4.15 The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1, J_2 = 4$) for $z = 6$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.16 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $z = 6$
Figure 4.17 The simulation result for the susceptibility $\chi(T)$ for the first kind of SW model ($J_1 = 1, J_2 = 4$) for $z = 6$
Figure 4.18  The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $z = 6$
Figure 4.19 The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $z = 7$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.20  The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1, J_2 = 4$) for $z = 8$

Note:  The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
The result for Binder 4th order cumulant $U_4$ is only plotted for $z = 7, 8$, shown in Figure 4.9 and Figure 4.10.

From these figures, the critical point $T_c$ shifts to high temperature for large $z$. For each $z$, the peak of the susceptibility $\chi$ and the specific heat $C_H$ shifts to higher temperature, approaching $T_c$ as the number of spins in the lattice, $N$, becomes large. The locations of these maxima are near that of the crossing point of the Binder 4th order cumulant $U_4$.

Table 4.1 Values of the critical temperature $T_c$ for the first kind of small-world model

<table>
<thead>
<tr>
<th>$J_1$</th>
<th>$J_2$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3.791</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5.391</td>
</tr>
</tbody>
</table>

In Figure 3.18, the plot of $T_c/z$ vs. $z$ from Monte Carlo simulations of several models, the points of the first kind of SW-model with $J_1 = 1, J_2 = 1$ fall closely to those of the $z$-model. Here, the $T_c$ values used are the crossing point of the Binder 4th order cumulant for larger size lattices, as described in Equation (2.159). But for the SW-model with larger $J_2$, for example, $J_2 = 4$, the $T_c/z$ values become very large as $z$ becomes large, which implies the interaction strength of small-world connections affects the critical point greatly. These can also be seen from the comparison plots for various $z$ for a large system size, $L = 256$, Figure 4.21 to Figure 4.28. For systems with same long-range interaction strength, the critical temperature shifts to higher temperature for systems with larger amount of
Figure 4.21 The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1, J_2 = 1$) for $L = 256$ for various $z$.

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.22 The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1, J_2 = 4$) for a large system size, $L = 256$, for $z = 5, 6, 7, 8$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
small world connections (\(z\) larger). Also for systems with same amount of long-range connections, stronger long-range interaction leads to higher critical temperature. Table 4.1 shows the critical temperature values for the 1st kind of small-world model for various \(z\) and \(J_2\).

With a large amount of small-world bonds (long-range interaction bonds), here each spin has 1 or more long-range small-world connections, systems are expected to have the mean field properties. Thus just as in the \(z\)-model, it is expected that the value of \(\frac{\partial U_4(t)}{\partial t}\) at the theoretical critical point for an infinite size mean field system, \(U_4 = 0.2705201\) \([9]\), is proportional to \(N^{\frac{1}{2}}\), which is suggested by Equation (3.13). From the log-log plots for \(-\frac{\partial U_4(t)}{\partial t}\) vs. \(N\) given in Figure 4.29 and Figure 4.30, it can be seen that the values of \(\frac{\partial U_4(t)}{\partial t}\) for different system sizes but same \(z\) and \(J_1, J_2\) fall closely to a straight line of slope \(\frac{1}{2}\). Linear-fitting is performed with respect to Equation (3.14) for \(\frac{\partial U_4(t)}{\partial t}\), the best-fit prefactors for various \(z, J_1\) and \(J_2\) are shown in Table 4.2.

<table>
<thead>
<tr>
<th>(J_1)</th>
<th>(J_2)</th>
<th>(z)</th>
<th>(a_0)</th>
<th>(N^{0.5})</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.038</td>
<td>0.040</td>
<td>0.027</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0.032</td>
<td>0.018</td>
<td>0.017</td>
</tr>
</tbody>
</table>

Figure 4.31 to Figure 4.34 are the plots for the Binder 4th order cumulant scaling for the first kind of SW-model with both \(J_2 = 1\) and \(J_2 = 4\). The scaling law used is the mean
Figure 4.23 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for a large system size, $L = 256$, for $z = 5, 6, 7, 8$. 
Figure 4.24 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1, J_2 = 4$) for a large system size, $L = 256$, for $z = 5, 6, 7, 8$.
Figure 4.25 The simulation result for the susceptibility $\chi_T$ for the first kind of SW model ($J_1 = 1, J_2 = 1$) for a large system size, $L = 256$, for $z = 5, 6, 7, 8$.
Figure 4.26 The simulation result for the susceptibility $\chi T$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for a large system size, $L = 256$, for $z = 5, 6, 7, 8$
Figure 4.27 The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for a large system size, $L = 256$, for $z = 5, 6, 7, 8$. 
Figure 4.28 The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1, J_2 = 4$) for a large system size, $L = 256$, for $z = 5, 6, 7, 8$. 
Figure 4.29  The log-log plot of $\frac{\partial U_4(t)}{\partial t}|_{u_\infty}$ vs. $N$ for the first kind of SW-model ($J_1 = 1, J_2 = 1$), for $z = 5, 6, 7, 8$

Note: Here $u_\infty = 0.2705201$, is the theoretical value for infinite size mean field models [9]. The line has a slope of $\frac{1}{2}$, which is the expected value.
Figure 4.30 The log-log plot of $-\frac{\partial U_4(t)}{\partial t}|_{u_\infty}$ vs. $N$ for the first kind of SW-model ($J_1 = 1, J_2 = 4$), for $z = 5, 6, 7, 8$.

Note: Here $u_\infty = 0.2705201$, is the theoretical value for infinite size mean field models [9]. The line has a slope of $\frac{1}{2}$, which is the expected value.
field scaling relation, Equation (3.12). In each plot (same $J_2$ and $J_1$), the points of different size systems fall together. The critical temperature has different values for systems with different $z$ and $J_2$, which are obtained from the crossing point of the Binder 4th order cumulant. From the plots, it can be seen that the larger the small-world interaction strength ($J_2 = 4$) is, the closer the $U_4$ value is to the theoretical value of an infinite size mean field system. Figure 4.35 is the $U_4$ scaling for a large system size $L = 256$ and long-range interaction strength $J_2 = 1$. Figure 4.36 is for $J_2 = 4$. These two plots show that for large size systems, the critical temperature of the first kind of small-world system approaches to the predicted value of an infinite size mean-field system, the larger the amount of small-world connections, the nearer the critical temperature is to the mean-field value. Also, Figure 4.37 shows that for various long-range interaction strengths, $J_2 = 1, 4$, various system sizes, $L = 128, 256$, and various number of small world connections, $z = 5, 6$, the $U_4$ scaling curves collapse together in a close region.

Again there is no scaling for $C_H$ without logarithmic corrections due to $\alpha = 0$ [9] [34]. The order parameter $|M|$ and the susceptibility $\chi$ mean field scaling laws, Equation (3.15) and Equation (3.16), are applied to the first kind of SW-model and satisfactory scaling results are obtained. Figure 4.38 and Figure 4.41 are the results for various system sizes for $z = 5$ and $J_2 = 1$. Figure 4.39 and Figure 4.40 are the results for the order parameter for a system size $L = 256$, and various nearest connections $z = 5, 6, 7, 8$ for $J_2 = 1$ and $J_2 = 4$. Figure 4.42 and Figure 4.43 are the scaling results for the susceptibility for a large system size $L = 256$, and various nearest connections $z = 5, 6, 7, 8$ for $J_2 = 1$ and $J_2 = 4$. 
Figure 4.31 The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model for $J_1 = 1, J_2 = 1, z = 5, U_4 = f_u(tN^{1/2})$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model for $J_1 = 1, J_2 = 1, z = 6, U_4 = f_u(tN^{1/2})$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.33  The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model for $J_1 = 1$, $J_2 = 4$, $z = 5$, $U_4 = f_u(tN^{1/2})$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.34  The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model for $J_1 = 1, J_2 = 4, z = 6, U_4 = f_u(t N^{1/2})$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.35  The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model for a large system size, $L = 256$, for $J_1 = 1$, $J_2 = 1$, $z = 5, 6, 7, 8$, $U_4 = f_u(tN^{\frac{1}{2}})$

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.36  The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model for a large system size, $L = 256$, for $J_1 = 1$, $J_2 = 4$, $z = 5, 6, 7, 8$, $U_4 = f_u(tN^{1/2})$

Note:  The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.37 The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model for various system sizes, long-range interactions and connections ($L = 128, 256$, $J_1 = 1$, $J_2 = 1, 4$, $z = 5, 6$, $U_4 = f_u(tN^{1/2})$)

Note: The exact value at criticality [9] of $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.38 The order parameter $|M|$ scaling for the first kind of SW-model ($J_1 = 1, J_2 = 1$) for $z = 5$

Note: Here the scaling relation is $|M| = N^{-\frac{1}{4}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. 
Figure 4.39  The order parameter $|M|$ scaling for the first kind of SW-model for a large system size, various long-range connections ($L = 256$, $J_1 = 1$, $J_2 = 1$, $z = 5, 6, 7, 8$)

Note: Here the scaling relation is $|M| = N^{-rac{1}{4}} f_M(tN^{1/2})$, thus $\beta = \frac{1}{2}$. 
Figure 4.40  The order parameter $|M|$ scaling for the first kind of SW-model for a large system size, various long-range connections ($L = 256$, $J_1 = 1$, $J_2 = 4$, $z = 5, 6, 7, 8$)

Note:  Here the scaling relation is $|M| = N^{-\frac{1}{4}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. 
Figure 4.41  The susceptibility $\chi$ scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 1$) for $z = 5$

Note: Here $\chi_{|M|} = N^{\frac{1}{2}} f_{\chi}(t N^{\frac{1}{z}})$, thus $\gamma = 1$. 
Figure 4.42 The susceptibility $\chi$ scaling for the first kind of SW-model for a large system size, various long-range connections ($L = 256$, $J_1 = 1$, $J_2 = 1$, $z = 5, 6, 7, 8$)

Note: Here the scaling relation is $\chi(\frac{tN}{1}) = N^{\frac{\gamma}{2}} f_\chi(tN^{\frac{1}{\gamma}})$, thus $\gamma = 1$. 
Figure 4.43 The susceptibility $\chi$ scaling for the first kind of SW-model for a large system size, various long-range connections ($L = 256, J_1 = 1, J_2 = 4, z = 5, 6, 7, 8$)

Note: Here the scaling relation is $\chi_{\{M\}} = N^{1/2} f_\chi(tN^{1/2})$, thus $\gamma = 1$. 
Similar to the analysis of the z-model in previous chapter, the exponents $\beta$ and $\gamma$ of the first kind of SW-model with strong small world bonds and $z \geq 5$ ($w \propto L^2$) have the mean field critical exponent values, $\beta = \frac{1}{2}$ and $\gamma = 1$. Thus it can be concluded that the first kind of SW-model with strong long-range interactions and $z \geq 5$ ($w \propto L^2$) scales in the same way as the z-model, which has mean field properties. For large $tN^{\frac{1}{2}}$ with $N \gg 1$ and $t \ll 1$, the log-log plots of scaling relations Equation (3.15) and Equation (3.16) reproduce the mean field values of critical exponents $\beta$ and $\gamma$, Figure 4.44 to Figure 4.52.

In Figure 4.44 to Figure 4.48, the branches below $T_c$ and above $T_c$ of $|M|$ approach the corresponding slope $\frac{1}{2}$ line and slope $-\frac{1}{2}$ line. Table 4.3 shows the best-fit values of the slope of these two branches (from $tN^{\frac{1}{2}} = 4$ to 12) for various system sizes and various $z$ values. For same $z$, the slope of the below $T_c$ branch approaches to $\frac{1}{2}$ as the system size is increased; and for same size systems, shown in the table are $L = 256$, the slope of the below $T_c$ branch is more closer to the expected value $\frac{1}{2}$ for larger $z$. That indicates larger size systems with a larger amount of long-range interactions exhibit stronger mean field properties.

In Figure 4.49 to Figure 4.52, the branches below $T_c$ of $\chi T$ approach the correspond slope $-1$ line.

The scaling curves for the Binder 4th cumulant for various large system sizes, $L = 128, 256$, various nearest connections $z = 5, 6, 7, 8$, and various interaction strengths, $J_2 = 1, J_2 = 4$ fall together, Figure 4.37. It is the same for the order parameter $|M|$ and the susceptibility, Figure 4.48 and Figure 4.52.
Figure 4.44  The order parameter $|M|$ scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 1$) for $z = 5$, log-log

Note: Here $|M| = N^{-\frac{1}{2}} f_M(t N^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 4.45  The order parameter $|M|$ scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 1$) for $z = 8$, log-log

Note: Here $|M| = N^{-\frac{1}{2}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 4.46 The order parameter $|M|$ scaling for the first kind of SW-model for a large system size, various long-range connections ($L = 256$, $J_1 = 1$, $J_2 = 1$, $z = 5, 6, 7, 8$)

Note: Here scaling relation is $|M| = N^{-\frac{1}{2}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 4.47  The order parameter $|M|$ scaling for the first kind of SW-model for a large system size, various long-range connections ($L = 256, J_1 = 1, J_2 = 4, z = 5, 6, 7, 8$)

Note: Here scaling relation is $|M| = N^{-\frac{1}{2}} f_M(t N^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 4.48 The order parameter $|M|$ scaling for the first kind of SW-model for various system sizes, long-range interactions and connections ($L = 128, 256$, $J_1 = 1, J_2 = 1, 4$, $z = 5, 6$).

Note: Here scaling relation is $|M| = N^{-\frac{1}{2}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 4.49  The susceptibility $\chi$ scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 1$) for $z = 5$, log-log

Note: Here $\chi_{|M|} = N^{\frac{1}{2}} f_\chi (t N^{\frac{1}{2}})$, thus $\gamma = 1$. The straight line has a slope of $-1$. 
Figure 4.50 The susceptibility $\chi$ scaling for the first kind of SW-model for a large system size, various long-range connections ($L = 256$, $J_1 = 1$, $J_2 = 1$, $z = 5, 6, 7, 8$)

Note: Here scaling relation is $\chi_M = N^{\frac{\gamma}{2}} f_N(tN^{\frac{1}{2}})$, thus $\gamma = 1$. The straight line has a slope of $-1$. 
Figure 4.51  The susceptibility $\chi$ scaling for the first kind of SW-model for a large system size, various long-range connections ($L = 256$, $J_1 = 1$, $J_2 = 4$, $z = 5, 6, 7, 8$)

Note: Here scaling relation is $\chi(M) = N^{\frac{1}{4}} f_{\chi}(t N^{\frac{1}{2}})$, thus $\gamma = 1$. The straight line has a slope of $-1$. 
The susceptibility $\chi$ scaling for the first kind of SW-model for various system sizes, long-range interactions and connections ($L = 128, 256, J_1 = 1, J_2 = 1, 4, z = 5, 6$)

Note: Here the scaling relation is $\chi|_M = N^{\frac{\gamma}{2}} f(x(tN^{1/2}))$, thus $\gamma = 1$. The straight line has a slope of $-1$. 
Table 4.3  Best-fit slopes of $|M|$ scaling ($tN^{\frac{1}{2}} = 4$ to $12$, $J_1 = J_2$, 1st SW)

<table>
<thead>
<tr>
<th>$z$</th>
<th>$L$</th>
<th>Branch $T &lt; T_c$</th>
<th>Branch $T &gt; T_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>32</td>
<td>0.264 ± 0.006</td>
<td>−0.431 ± 0.003</td>
</tr>
<tr>
<td>5</td>
<td>64</td>
<td>0.375 ± 0.006</td>
<td>−0.454 ± 0.002</td>
</tr>
<tr>
<td>5</td>
<td>128</td>
<td>0.447 ± 0.003</td>
<td>−0.468 ± 0.007</td>
</tr>
<tr>
<td>5</td>
<td>256</td>
<td>0.480 ± 0.002</td>
<td>−0.471 ± 0.008</td>
</tr>
<tr>
<td>8</td>
<td>32</td>
<td>0.333 ± 0.004</td>
<td>−0.410 ± 0.001</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>0.422 ± 0.003</td>
<td>−0.441 ± 0.002</td>
</tr>
<tr>
<td>8</td>
<td>128</td>
<td>0.466 ± 0.003</td>
<td>−0.459 ± 0.004</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>0.493 ± 0.002</td>
<td>−0.466 ± 0.006</td>
</tr>
<tr>
<td>5</td>
<td>256</td>
<td>0.480 ± 0.002</td>
<td>−0.471 ± 0.008</td>
</tr>
<tr>
<td>6</td>
<td>256</td>
<td>0.484 ± 0.001</td>
<td>−0.482 ± 0.007</td>
</tr>
<tr>
<td>7</td>
<td>256</td>
<td>0.493 ± 0.003</td>
<td>−0.469 ± 0.008</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>0.493 ± 0.002</td>
<td>−0.466 ± 0.006</td>
</tr>
</tbody>
</table>

From above analysis, the scalings for $U_4$, $|M|$ and $\chi$ for the first kind of small-world model with strong long-range interactions ($w \propto L^2$) and large amounts of long-range connections ($z \geq 5$) obey the same scaling laws as for the $z$-model, which has mean field critical behavior. Also the critical exponent $\beta$ and $\gamma$ have mean field values, $\frac{1}{2}$ and 1. It can be concluded that the first kind of small-world model with non-vanishing density strong long-range interactions has mean field properties in the vicinity of the critical point $T_c$. $T_c$ approaches to higher value when the long-range interaction strength or the amount of long-range connections is increased.
4.3.2 Vanishing density strong small world connections with \( w \leq \frac{L^2}{2} \) and \( w \propto L \)

In the previous section, it is known that magnetic systems with large amounts of small-world connections (each spin has one or more small-world connections) and strong long-range interaction have mean field behavior. But how does a small amount of long-range connections affect the critical properties? Hastings presented a modified small-world model and predicted that a small world system with long-range connections added with probability \( p \) is equivalent to a system in which each site in the lattice coupling to every other site with a strength of \( \frac{p}{V} \), \( V \) is the total sites in the lattice [17]. He concluded that such systems (with small amount of long-range connections) will exhibit mean field behavior. In this section, systems with a small amount of small world connections are studied to see the effects of vanishing density strong long-range connections.

Here, the first kind of small-world model with strong long-range interaction is simulated for systems which have \( w = L (z \leq 5) \) small-world connections.

The first kind of small world Ising systems studied here have \( w \) small-world connections. On average, each spin has \( z \) nearest neighbor connections. From Equation (2.165), \( z = 4 + \frac{2w}{L^2} = 4(1 + \phi) \) and \( \phi = \frac{w}{2L^2} \). Thus, each spin has \( 4\phi \) long-range small-world connections. When system size, \( L \), is increased, \( \phi \) approaches to zero, thus these systems have vanishing long-range connection density. The interaction strengths are set to be \( J_1 = 1 \), \( J_2 = 1 \) and \( J_1 = 1, J_2 = 4 \). The system with no small world connections added, \( w = 0 \), so \( \phi = 0 \) and \( z = 4 \), degenerates to the pure square lattice system. Simulations are done for
systems with $w = L$ long-range connections and pure square lattice Ising systems. Results are shown in Figure 4.53 to Figure 4.64.

The simulation results show that for pure square lattice, the critical point falls at the expected value $T_c = 2.269 \cdots$. For systems which have $w = L$ small-world connections, the critical point $T_c$ shifts to high temperature for stronger interactions. As the system size increases, the curves of the Binder 4th order cumulant $U_4$, the order parameter $|M|$, the susceptibility $\chi$, and the specific heat $C$ approach to the pure lattice Ising curves, Figure 4.53 to Figure 4.64. In Figure 4.58, Figure 4.62, Figure 4.59, Figure 4.63, the pure lattice values of $|M|$ and $C$ are shown, thus it is easy to see that the curves for large $L$ systems with $w = L$ small world connections are very close to the pure lattice curves. Also, the peaks of the susceptibility $\chi$ and the specific heat $C_H$ shift to lower temperature, approaching $T_c$. These peaks are near the crossing point of the Binder 4th order cumulant $U_4$.

It can be seen from the simulation results that with $w = L$ small-world bonds, systems have critical point near that of a pure square lattice. The scaling results of the Binder 4th order cumulant $U_4$, the order parameter $|M|$ and the susceptibility $\chi$, which are according to Ising scaling relations, are shown in Figure 4.65 to Figure 4.70. Here, the Ising scaling exponents, $\beta = \frac{1}{8}$, $\gamma = \frac{7}{4}$ and $\nu = 1$, are applied to Equation (2.156), Equation (2.148), and Equation (2.149). The critical temperature $T_c$ is affected by the long-range interaction strength, $J_2$, strong interaction strength leads to high critical temperature.
Figure 4.53  The simulation result for the Binder 4th order cumulant $U_4$ for the pure square lattice

Note: The exact value at criticality [31] of $U_4 = 0.615$ is given by the horizontal line.
Figure 4.54  The simulation result for the order parameter $|M|$ for the pure square lattice
Figure 4.55  The simulation result for the specific heat $C_H$ for the pure square lattice
Figure 4.56 The simulation result for the susceptibility $\chi T$ for the pure square lattice
Figure 4.57  The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1, J_2 = 1$) for $w = L$

Note:  The exact value at criticality [31] of $U_4 = 0.615$ is given by the horizontal line.
Figure 4.58 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $w = L$. 
Figure 4.59 The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $w = L$. 
Figure 4.60 The simulation result for the susceptibility $\chi^T$ for the first kind of SW model ($J_1 = 1$, $J_2 = 1$) for $w = L$
Figure 4.61 The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $w = L$.

Note: The exact value at criticality [31] of $U_4 = 0.615$ is given by the horizontal line.
Figure 4.62 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $w = L$. 
Figure 4.63 The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1$, $J_2 = 4$) for $w = L$
Figure 4.64 The simulation result for the susceptibility $\chi T$ for the first kind of SW model ($J_1 = 1, J_2 = 4$) for $w = L$. 
Figure 4.65  The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model for $J_1 = 1$, $J_2 = 1$ for $w = L$

Note: Here the scaling relation is $U_4 = U_4(tL^{1/\nu})$, where $\nu = 1$. The exact value at criticality [31] of $U_4 = 0.615$ is given by the horizontal line.
Figure 4.66  The Binder 4th order cumulant $U_4$ scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 4$) for $w = L$

Note: Here the scaling relation is $U_4 = U_4(tL^{\nu})$, where $\nu = 1$. The exact value at criticality [31] of $U_4 = 0.615$ is given by the horizontal line.
The order parameter $|M|$ scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 1$) for $w = L$, log-log

Note: The scaling relation is $|M| = L^{-\frac{\beta}{\nu}} f_M(t L^{\frac{1}{\nu}})$, where $\beta = \frac{1}{8}$, $\nu = 1$. The straight lines have slopes of $\frac{1}{8}$ and $-\frac{7}{8}$. 

Figure 4.67
The order parameter $|M|$ scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 4$) for $w = L$, log-log

Note: The scaling relation is $|M| = L^{-\frac{\beta}{\nu}} f_M(t L^{\frac{1}{\nu}})$, where $\beta = \frac{1}{8}$, $\nu = 1$. The straight lines have slopes of $\frac{1}{8}$ and $-\frac{7}{8}$. 
Figure 4.69  The susceptibility $\chi$ scaling for the first kind of SW-model ($J_1 = 1, J_2 = 1$) for $w = L$, log-log

Note:  The scaling relation is $\chi_{|M|} = L^{\frac{\gamma}{\nu}} f_x(t L^{\frac{1}{\nu}})$, where $\gamma = 1.75$, $\nu = 1$. The straight line has a slope of $-1.75$. 
Figure 4.70  The susceptibility $\chi$ scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 4$) for $w = L$, log-log

Note: The scaling relation is $\chi_{|M|} = L^{\gamma} f_x (tL^{-\nu})$, where $\gamma = 1.75$, $\nu = 1$. The straight line has a slope of $-1.75$. 
Compare Figure 4.67 to Figure 4.68, and Figure 4.69 to Figure 4.70, it can be seen, for the first kind of SW-model with a small amount of small-world connections, \( w = L \), when scaled with the Ising relation and Ising exponent values \( \beta = \frac{1}{8}, \gamma = \frac{7}{4} \) and \( \nu = 1 \), the scaled curves of different system sizes with moderate long-range interaction \( (J_2 = 1) \) fall closer than that with stronger small-world interaction \( (J_2 = 4) \). For large \( tL^{\frac{1}{\nu}} \) with \( L \gg 1 \) and \( t \ll 1 \), the plots of the order parameter scaling, Figure 4.67 and Figure 4.68, also show that the branches below \( T_c \) approach to the corresponding slope \( \frac{1}{8} \) line and the branches above \( T_c \) approach to the corresponding slope \( -\frac{7}{8} \) line. For systems with stronger interaction, scaling points scatter away from the corresponding lines. Also, in the plots of the susceptibility scaling, Figure 4.69 and Figure 4.70, the branches below \( T_c \) approach the corresponding slope \( -1.75 \) line.

From these simulation results and analysis, it can be concluded that even a small amount of long-range interactions (small-world connections) can affect the critical behavior of a system. Systems with vanishing density strong long-range connections still have Ising behavior. Here, here systems have \( w = L \) SW connections and \( \phi = \frac{w}{2L^2} \) approaches to zero as \( L \) is increased. The mean field effect that Hastings predicted in \[17\] is not observed. As the small-world interaction becomes stronger, a small-lattice effect similar to the mean field effect is seen before the Ising values. The critical point approaches the theoretical Ising value from high temperature side as the system size increases. When the strength of long-range interaction increases, the effective critical temperature for a fixed system size also shifts to higher values.
4.3.3 Weak small world connections with $w = \frac{L^2}{L}$

Magnetic Ising systems with strong long-range interaction strength present Ising critical behavior if they have non-vanishing density small-world connections and mean field behavior if they have vanishing density small-world connections. But how does weak long-range interaction affect the critical behavior? Hastings predicted mean field behavior for his long-range model [17], in which each site of the lattice interacts with any other site with a weak strength. He argued that with a large amount of long-range interactions, a system can be approximately solved by mean-field theory, thus the system exhibits mean field behavior even the long-range interaction is very weak. So that it is expected a small world Ising system will exhibit mean field behavior if all spins in the system have a long-range weak interaction. In this section, the effects of weak long-range interaction are studied for the first kind of small world model which has $w = \frac{L^2}{L}$ small-world connections, or say each spin has one long-range connection.

The Monte Carlo simulation is performed for the first kind of small-world systems with $J_1 = 1$, $J_2 = 0.01$ and $J_1 = 1$, $J_2 = 0.05$. The simulation results are shown in Figure 4.71 to Figure 4.78. For very weak long-range interaction $J_2 = 0.01$, the crossing point of $U_4$ is near the critical point of a pure square lattice Ising model, and the critical temperature is a little higher than $T_c = 2.269 \cdots$. As the long-range interaction becomes stronger, the critical temperature becomes higher and the crossing point shifts to the mean field critical point for infinite size system (the point at where $U_4 = 0.2705401$). The peaks of the susceptibility $\chi$ and the specific heat $C_H$ shift to higher temperature correspondingly.
Figure 4.71 The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1, J_2 = 0.01$) for $w = \frac{L^2}{2}$.

Note: The exact values at Ising criticality [31] of $U_4 = 0.615$ and at the mean field criticality [9] $U_4 = 0.2705201$ are given by the horizontal lines.
Figure 4.72 The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1$, $J_2 = 0.01$) for $w = \frac{L^2}{2}$.
Figure 4.73  The simulation result for the specific heat $C_H$ for the first kind of SW model $(J_1 = 1, J_2 = 0.01)$ for $w = \frac{L^2}{2}$
Figure 4.74 The simulation result for the susceptibility $\chi T$ for the first kind of SW model ($J_1 = 1, J_2 = 0.01$) for $w = \frac{L^2}{2}$
Figure 4.75  The simulation result for the Binder 4th order cumulant $U_4$ for the first kind of SW model ($J_1 = 1$, $J_2 = 0.05$) for $w = \frac{L^2}{2}$

Note: The exact values at Ising criticality [31] of $U_4 = 0.615$ and at the mean field criticality [9] $U_4 = 0.2705201$ are given by the horizontal lines.
Figure 4.76  The simulation result for the order parameter $|M|$ for the first kind of SW model ($J_1 = 1$, $J_2 = 0.05$) for $w = \frac{L^2}{2}$.
Figure 4.77 The simulation result for the specific heat $C_H$ for the first kind of SW model ($J_1 = 1, J_2 = 0.05$) for $w = \frac{L^2}{2}$.
Figure 4.78 The simulation result for the susceptibility $\chi T$ for the first kind of SW model ($J_1 = 1$, $J_2 = 0.05$) for $w = \frac{L^2}{2}$.
From Figure 4.71, for very weak long-range interactions \( J_2 = 0.01 \), the critical point falls to the higher side and near the critical value of a pure square lattice, \( T_c = 2.269 \ldots \). Since the long-range interaction is very weak and the critical point is near that of a pure square lattice, Ising scaling relations are first used. The critical temperature is set to be \( T_c \approx 2.3348 \), which is the best-fit value for the scaling for the Binder 4th order cumulant \( U_4 \), the order parameter \( |M| \) and the susceptibility \( \chi \) (eye-observe). The Ising-like scaling results for the Binder 4th order cumulant \( U_4 \), the order parameter \( |M| \), and the susceptibility \( \chi \) are shown in Figure 4.79 to Figure 4.81. In the log-log plots of the order parameter \( |M| \) and the susceptibility \( \chi \), the points fall in a close region near the corresponding slope line. The below \( T_c \) branches of the order parameter \( |M| \) collapse together, but the above \( T_c \) branches do not. As \( tL^{\frac{1}{2}} \) goes larger, the curves of different system sizes separate a little away. As the system size increased, the \( U_4 \) crossing value approaches to the mean field value for an infinite large system, also see Figure 4.71. This suggests the mean field behavior may exist. Then the mean field scaling relations are used to scaling the Binder 4th order cumulant \( U_4 \), the order parameter \( |M| \) and the susceptibility \( \chi \). Here, the critical temperature used is not the crossing point of \( U_4 \) for large size systems, but the temperature at where \( U_4 = 0.2705201 \), the theoretical value for an infinite size mean field system. Here \( T_c = 2.358 \). Scaling results are shown in Figure 4.82, Figure 4.83, and Figure 4.84. Compared to the corresponding Ising-like scaling plots, it can be seen, although the mean field behavior appears, the Ising-like behavior is stronger. But from the \( U_4 \) scaling, Figure 4.79
and Figure 4.85, it shows that large size systems exhibit mean-field like than Ising-like behavior.

Figure 4.85 and Figure 4.86 are the Ising-like scaling results for the Binder 4th order cumulant $U_4$ and the order parameter $|M|$ for $J_2 = 0.05$. The critical temperature used is better-fitting value. It turns out that $T_c$ has different values for $U_4$ scaling and $|M|$ scaling. Although the scaling result of the order parameter $|M|$ looks good, no valuable results are obtained when the Ising scaling relation is applied to scale the susceptibility $\chi$. Also, it can be seen from Figure 4.75, the $U_4$ crossing point is far away from the Ising value, it approaches to the mean field value for larger system sizes. Figure 4.87 to Figure 4.89 are the mean field scaling results, compared to the Ising-like scaling, Figure 4.85 to Figure 4.86, these results are much better. Thus, when the long-range interaction becomes stronger, the Ising-like properties are very weak and the small world systems studied here behave mean field like.

Consider systems with a large size, $L = 384$, but with different long-range interactions, $J_2 = 0.01, 0.05, 0.1, 0.5$, the mean field scaling relations are applied since even very weak long range interaction, $J_2$, affects the system to have the mean field behavior. Figure 4.90 to Figure 4.92 show the scaling results. The curves for the order parameter $|M|$ and the susceptibility $\chi$ for weak small-world connection, $J_2 = 0.01$, are not far away from that of the stronger ($J_2 = 0.05, 0.1, 0.5$) long-range interaction systems in the log-log plots, Figure 4.91 and Figure 4.92. These curves approach to the straight line of the slope for mean field system. Also, in the $U_4$ scaling plot, Figure 4.90, the curves for different long-
Figure 4.79 The the Binder 4th order cumulant $U_4$ Ising-like scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 0.01$) for $w = L^2/2$.

Note: The scaling relation is $U_4 = U_4(t L^{1/\nu})$, where $\nu = 1$. The exact values at Ising criticality [31] of $U_4 = 0.615$ and at the mean field criticality [9] $U_4 = 0.2705201$ are given by the horizontal lines.
The order parameter $|M|$ Ising-like scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 0.01$) for $w = L^2/2$, log-log

Note: The scaling relation is $|M| = L^{-\beta/\nu} f_M(tL^{1/\nu})$, where $\beta = \frac{1}{8}$, $\nu = 1$. The straight lines have slopes of $\frac{1}{8}$ and $-\frac{7}{8}$. 

Figure 4.80

$T_c = 2.3348$

slope = 0.125

slope = -0.875

$J_1 = 1, J_2 = 0.01, L^2/2$ sw cons

$|M|^{\beta/\nu}$

$N = 16 \times 16$

$N = 32 \times 32$

$N = 64 \times 64$

$N = 128 \times 128$

$N = 256 \times 256$

$N = 384 \times 384$

$x = tL^{1/\nu} (\nu = 1.0)$
Figure 4.81 The susceptibility $\chi$ Ising-like scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 0.01$) for $w = \frac{L^2}{2}$, log-log

Note: The scaling relation is $\chi_{|M|} = \frac{L^{\gamma}}{\nu} f_{\chi}(tL)\frac{1}{2}$, where $\gamma = 1.75$, $\nu = 1$. The straight line has a slope of $-1.75$. 
Figure 4.82  The Binder 4th order cumulant $U_4$ mean field like scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 0.01$, $z = 5$, $U_4 = f_u(tN^{1/2})$)

Note:  The exact value at the mean field criticality [9] $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.83  The order parameter $|M|$ mean field like scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 0.01$) for $z = 5$

Note:  Here $|M| = N^{-\frac{1}{4}} f_M(tN^{\frac{4}{7}})$, and $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$.  

Figure 4.84  The susceptibility $\chi$ mean field like scaling for the first kind of SW-model $(J_1 = 1, J_2 = 0.01)$ for $z = 5$

Note: Here $\chi_{|M|} = N^{\frac{1}{2}} f_\chi(t N^{\frac{1}{2}})$, and $\gamma = 1$. The straight line has a slope of $-1$. 
Figure 4.85 The $U_4$ Ising-like scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 0.05$) for $w = \frac{L^2}{2}$.

Note: The scaling relation is $U_4 = U_4(tL^{1/\nu})$, where $\nu = 1$. The exact values at Ising criticality [31] of $U_4 = 0.615$ and at the mean field criticality [9] $U_4 = 0.2705201$ are given by the horizontal lines.
The order parameter $|M|$ Ising-like scaling for the first kind of SW-model $(J_1 = 1, J_2 = 0.05)$ for $\omega = L^2/2$, log-log

**Note:** Here the scaling relation is $|M| = L^{-\frac{\beta}{\nu}} f_M(tL^{\frac{1}{\nu}})$, where $\beta = \frac{1}{8}$, $\nu = 1$. The straight lines have slopes of $\frac{1}{8}$ and $-\frac{7}{8}$. 

Figure 4.86: The order parameter $|M|$ Ising-like scaling for the first kind of SW-model $(J_1 = 1, J_2 = 0.05)$ for $\omega = L^2/2$, log-log
Figure 4.87 The Binder 4th order cumulant $U_4$ mean field like scaling for the first kind of SW-model ($J_1 = 1, J_2 = 0.05$) for $z = 5$, $U_4 = f_u(tN^{1/2})$

Note: The exact value at the mean field criticality [9] $U_4 = 0.2705201$ is given by the horizontal line.
Figure 4.88  The order parameter $|M|$ mean field like scaling for the first kind of SW-model ($J_1 = 1$, $J_2 = 0.05$) for $z = 5$

Note: Here $|M| = N^{-\frac{1}{4}} f_M(tN^{\frac{1}{4}})$, and $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 4.89  The susceptibility $\chi$ mean field like scaling for the first kind of SW-model $(J_1 = 1, J_2 = 0.05)$ for $z = 5$

Note:  Here $\chi_{|M|} = N^{\frac{1}{2}} f_{\chi}(tN^{\frac{1}{2}})$, and $\gamma = 1$. The straight line has a slope of $-1$. 
range interaction systems fall in a closer region, even the weakest one, for $J_2 = 0.01$, is not far away from others.

From the above analysis, weak long-range interaction affect the first kind of small-world Ising model to have critical behavior if the number of long-range connections is large (so that each spin has more than one long-range connection(s)). With very weak long-range small-world connections, systems mostly behave Ising-like, but the mean field like behavior appears. As the small-world interaction becomes stronger, the critical temperature shifts to higher temperature and the system performs a changing from Ising-like to mean field like. Further more, for any fixed small value of $J_2$, the scaling crosses over from Ising-like to mean-field like.

4.3.4 Crossover analysis of the first kind of small-world model

Magnetic systems with a large amount of strong small-world connections have mean field behavior, while with a small amount of strong small-world connections such that the long-range connection density is vanishing and small system sizes behave more Ising-like. When the long-range interaction is very weak, systems behave Ising-like, but mean field properties are shown if the long-range connection density is non-vanishing. The crossover from Ising-like to mean field critical behavior in a small-world system is much more complicated, it is affected by the small-world interaction strength, range and the number of small-world connections.
Figure 4.90  The Binder 4th order cumulant $U_4$ mean field like scaling for the first kind of SW-model for a large system size, $L = 384$, and weak bonds. For $z = 5$, $J_1 = 1$, $J_2 = 0.01, 0.05, 0.1, 0.5$, and $U_4 = f_u(t N^{1/2})$

Note:  The exact value at the mean field criticality [9] $U_4 = 0.2705201$ is given by the horizontal line.
The order parameter $|M|$ mean field like scaling for the first kind of SW-model for a large system size, $L = 384$, and weak bonds ($z = 5$, $J_1 = 1$, $J_2 = 0.01, 0.05, 0.1, 0.5$)

Note: Here scaling relation is $|M| = N^{-\frac{1}{2}} f_M(t N^{\frac{1}{2}})$, and $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
The susceptibility $\chi$ mean field like scaling for the first kind of SW-model for a large system size, $L = 384$, and weak bonds ($z = 5$, $J_1 = 1$, $J_2 = 0.01, 0.05, 0.1, 0.5$)

Note: Here the scaling relation is $\chi|M| = N^{\frac{1}{2}}f_\chi(tN^{\frac{1}{2}})$, and $\gamma = 1$. The straight line has a slope of $-1$. 
The crossover from Ising-like to mean field critical behavior near the critical temperature $T_c$ occurs in many thermodynamic systems [15]. There are many discussions about the crossover caused by the interaction range $R$ [7] [46] [41] [43] [42].

Hastings constructed a small-world model by adding a weak coupling to every site in the lattice system instead of adding long-range links and analyzed the mean field behavior of the model [17] by using field theory. He pointed out that for weak long-range interaction small-world systems, the critical point shifts away from the Ising critical point, and is given by

$$\tilde{T}_c - T_c = (A_\chi p)^{1/\gamma}.$$  \hspace{1cm} (4.5)

Here, $T_c$ is the critical temperature of the pure square lattice Ising system, $\tilde{T}_c$ is the critical temperature of the weak small-world interaction system, $A_\chi$ is the prefactor in the scaling of the susceptibility $\chi$, and $p$ is the strength ratio of the long-range interaction to the short-range interaction. Below the critical point, the magnetization obtained by Hastings is given by [17]

$$M = \left(\tilde{T}_c - T\right)^{1/2} A_\chi^+ \left(\frac{\gamma A_\chi^+}{B_\chi}\right)^{1/2} \left(\tilde{T}_c - T_c\right)^{\beta - 1/2},$$  \hspace{1cm} (4.6)

where for systems presented in this dissertation $\beta = \frac{1}{8}$.

Compared to the Hastings result for small-world systems, in the first kind of small-world model presented in this thesis, the interaction ratio $p$ is,

$$p = \frac{J_2}{J_1}.$$  \hspace{1cm} (4.7)
The critical point is related to the Ising value as, for $J_1 = 1$,

$$\bar{T}_c - T_c = A_t J_1^{1/\gamma}, \quad (4.8)$$

where $A_t$ is a prefactor. The order parameter $|M|$ is suggested to be,

$$|M| = A_m \bar{T}_c^{1/2} \left(1 - \frac{T}{T_c}\right)^{1/2} \left(\bar{T}_c - T_c\right)^{\beta - \frac{1}{2}}. \quad (4.9)$$

Here $A_m$ is a prefactor.

To analyze the effects of long-range interactions on the crossover for the first kind of small-world model, Monte Carlo simulations are performed also for $J_2 = 0.1$ and $J_2 = 0.5$ with $w = \frac{t^2}{T}$ small-world connections in the system. Figure 4.93 shows the Binder 4th order cumulant as a function of $t$ for the very weak long-range interactions $J_2 = 0.01$, and Figure 4.94 shows the order parameter as a function of $t$. In the plot of $U_4$ vs. $t$, the crossing point of large size systems is very close to the Ising critical point. While, in the plot of $|M|$ vs. $t$, the order parameter points from large size systems fall apart from the Ising curve. Figure 4.95 shows the order parameter $|M|$ as a function of $t$ for $L = 384$ systems with different long-range interaction strength. When $J_2$ becomes stronger, the order parameter curve shifts away from the Ising curve to the mean field curve.

To see how the critical point changes with the changing of the long-range interaction strength, the modified $\bar{T}_c - T_c$ vs. $p$ relation Equation (4.8) from Hastings [17] is applied for the large size ($L = 384$) systems. The scaling result is shown in Figure 4.96. Its fitting to the exponent $\frac{1}{\nu}$ curve looks very good. Equation (4.9) is used to do the order parameter scaling to see whether the random small-world connection can be treated as the
Figure 4.93 The $U_4$ vs. $t$ plot for the first kind of SW model with weak long-range interaction connections ($J_1 = 1, J_2 = 0.01, w = \frac{L^2}{2}$)

Note: The exact values at mean field criticality [9] point of $U_4 = 0.2705201$ and Ising criticality [31] point of $U_4 = 0.615$ are given by the horizontal lines.
Figure 4.94 The \(|M|\) vs. \(t\) plot for the first kind of SW model with weak long-range interaction connections \((J_1 = 1, J_2 = 0.01, w = \frac{L^2}{2})\)
Figure 4.95  The comparison of the order parameter $|M|$ for different long-range interactions for the first kind of SW model for a large system size, $L = 384$, and $w = \frac{L^2}{2}$.

Note: The exact $|M|$ vs. $t$ for 2-dimensional Ising model and mean field theory are shown for analysis. As the long-range strength becomes stronger, the $|M|$ curve shifts away from the Ising curve to the mean field curve.
Figure 4.96  The relation between the critical temperature and long-range interaction strength for the first kind of small-world model for a large system size, $L = 384$, and $w = \frac{L^2}{2}$

Note: Here, each spin has a long-range connection. The plot shows that the critical temperature shifts away from the Ising critical temperature with an exponent relation to long interaction strength. $\overline{T}_c - T_c = A_t J_2^{1/\nu}$. The mean field effect on the order parameter $|M|$ for the first kind of SW model near the critical point is shown.
average weak field applied to all the spins in the system. The result is shown in Figure 4.97.

The scaling relation is,

$$|M|\tilde{T}^{-\frac{1}{2}}(\tilde{T}_c - T_c)^{-(\beta - \frac{1}{2})} = f_{w|M}(1 - \frac{T}{\tilde{T}_c}).$$

(4.10)

If the order parameter has a $\frac{1}{2}$ power relation with $1 - \frac{T}{\tilde{T}_c}$, the branch below $\tilde{T}_c$ should asymptote to a slope $\frac{1}{2}$ line. But the log-log plot shows that the branch below $\tilde{T}_c$ approaches some straight line with a slope between $\frac{1}{8}$ and $\frac{1}{4}$. That suggests that perhaps the small-world connections may not be approximated by a weak field applied to all the spins in the system.

Mon and Binder discussed the finite-size scaling and the crossover in the two-dimensional Ising model [46] and proposed a formula for the crossover scaling. In their work, the crossover scaling variable is chosen as,

$$g = t^{\frac{4-d}{2}}R^d.$$  

(4.11)

Where $R$ is the effective interaction range. This leads to the order parameter crossover finite-size scaling,

$$g = LR^{-\frac{4}{d+2}}.$$  

(4.12)

$$M_L = L^{-\frac{d}{2}}f_{Mcross}(LR^{-\frac{4}{d+2}}).$$  

(4.13)

In 2-dimensional Ising finite-size scaling, the above relations are

$$g = LR^{-2}.$$  

(4.14)
Figure 4.97 The order parameter scaling with Hastings’s weak field relation.
\[ M_L = L^{-\frac{1}{2}} f_{M\text{cross}}(LR^{-2}). \] (4.15)

Luijten, etc. studied the effects of the interaction shape and proposed the calculation of \( R \) [42] [39],
\[ R^2 = \frac{\sum_{i \neq j} |r_i - r_j|^2 J_{ij}}{\sum_{i \neq j} J_{ij}}, \] (4.16)
where \( J_{ij} \) is the interaction strength between spin \( i \) and \( j \), which are located at \( r_i \) and \( r_j \).

This equation is applied to the calculating of the effective range \( R \) to do the crossover scaling for the first kind of small-world model. When the average separation of any two spins on a square lattice is used as the average small-world bond length in calculating the effective interaction range, the scaling results are not acceptable. This may be because the effective range \( R \) should not be calculated from the average length of a square lattice in the case that both regular and small-world interaction exist. By assuming that the average effective distance between any two spins in the first kind of small-world model is the average length of a corresponding Bethe lattice, and from the above equation Equation (4.16), the effective range \( R \) can be calculated as
\[ R^2 = \frac{2NJ_1 + J_{2w}}{2NJ_1 + J_{2w}} \left( \frac{lnV}{ln(z-1)} \right)^2. \] (4.17)

Here, \( z \) is the effective number of neighbors. This relation is applied to the crossover scaling of the order parameter, the result is shown in Figure 4.98 and Figure 4.99. In Figure 4.98, \( z = 5 \), which is the actual number of neighbors, is used. And in Figure 4.99, \( z = 2.5 \), which is the number from the best-fitting (eye-observe) is used. It is not apparent
that there is any large difference in these two plots since the fluctuations may be large. In both of these scaling results, the points fall in a close region but not on a single curve.

In this section, the finite-size crossover scaling is tried for the first kind of small-world model to study the crossover from the Ising-like to the mean field critical behavior. The scaling of the order parameter suggests not to treating the small-world interaction as a weak field applied to all the spins in the system. More effective methods need to be found to calculate the effective interaction range.

### 4.4 Simulation results and analysis of the second kind of small-world model

Novotny and Wheeler discussed the possibility of a material related to small-world networks and the critical properties of one-dimensional small-world systems with small-world bonds built from atoms [53]. The second kind of small-world model presented here is similar to the model in their work, except that their model is based on a one-dimensional spin chain while the second kind of small-world model presented here is based on a two-dimensional square lattice.

This part of work concentrates on the small-world effect of spin-chains, so it is assumed that the interaction strengths between nearest neighbor spins are the same. Thus, $J_1 = 1$ and $J_3 = 1$ are used in the Hamiltonian Equation (4.2). The systems simulated for the second kind of small-world model have $w = L$ and $w = \frac{L^2}{2}$ small-world connections
The crossover scaling for the first kind of SW model with logarithmic length of Bethe lattice used as the effective small-world interaction range.

Note: In this scaling, the effective small-world interaction range used is the logarithmic length of a Bethe lattice with $z = 5$. 
Figure 4.99  The crossover scaling for the first kind of SW model with modified logarithmic length used as the effective small-world interaction range.

Note:  In this scaling, the effective small-world interaction range used is the logarithmic length of a Bethe lattice with $z = 2.5$. 
built from spin chains. The Monte Carlo simulation for systems with $w = L$ small-world connections is performed for length series

$$L = 16, \ 32, \ 64, \ 128, \ 256.$$  \hfill (4.18)

so the total numbers of regular-lattice spins are

$$N = L^2 = 256, \ 1024, \ 4096, \ 16384, \ 65536.$$  \hfill (4.19)

And for systems which have $w = \frac{L^2}{2}$ small-world connections, Monte Carlo simulation is performed for length series

$$L = 16, \ 32, \ 64, \ 96,$$  \hfill (4.20)

thus the total numbers of regular-lattice spins are

$$N = L^2 = 256, \ 1024, \ 4096, \ 9216.$$  \hfill (4.21)

Because the small-world short-cuts are randomly added, the summation of the short-cut lengths in the systems is different for each run. Thus, the total spins in a system is not a constant in different runs. The fluctuation of the total spins in a system is assumed to be small and the Monte Carlo simulation is performed for only one sample for each system size.

Figure 4.100 to Figure 4.103 are the simulation results for the order parameter $|M|$, the Binder 4th order cumulant $U_4$, the isothermal susceptibility times the temperature $\chi T$, and the isomagnetic field specific heat $C_H$ for the second kind of small-world Ising systems.
with \( w = L \) small-world connections. From Figure 4.100, it can be seen, the critical temperature \( T_c \), which is from the crossing point of the Binder 4th order cumulant \( U_4 \), is very close to that of a pure square lattice Ising system. Also, In Figure 4.102 and Figure 4.103, the peaks of the susceptibility \( \chi \) and the specific heat \( C_H \) shift to the Ising value of \( T_c \) as the lattice system size \( L \) becomes large.

The critical point of the second kind of small-world Ising system with \( w = L \) small-world bonds is very close to that of a pure square lattice Ising system, \( T_c = 2.269 \cdots \). Such systems may behave like pure square lattice Ising systems. The finite size scaling of the Binder 4th order cumulant \( U_4 \), the order parameter \( |M| \), and the isothermal susceptibility times the temperature \( \chi T \) are analyzed by using the scaling relations for the pure square lattice Equation (2.156), Equation (2.148), and Equation (2.149). The critical exponents values used are Ising values \( \beta = \frac{1}{8}, \gamma = \frac{7}{4} \) and \( \nu = 1 \). In these scaling plots, the scaled curves for systems of different sizes fall together. For large \( tL^{\frac{1}{\nu}} \) with \( L \gg 1 \) and \( t \ll 1 \), the branches below \( T_c \) of \( |M| \) and \( \chi T \) approach the corresponding slopes \( \frac{1}{8} \) and \(-\frac{7}{4}\) lines (log-log plot).

Now consider the second kind of small-world Ising systems with \( w = \frac{L^2}{2} \) bonds. The simulation results for the order parameter \( |M| \), the Binder 4th order cumulant \( U_4 \), the isothermal susceptibility times the temperature \( \chi T \), and the isomagnetic field specific heat \( C_H \) are shown in Figure 4.107 to Figure 4.110. Figure 4.107 shows that the crossing point of the Binder 4th order cumulant \( U_4 \), which is used to determined the critical temperature \( T_c \), is also very close to that of a pure square lattice Ising system, just like that of the
Figure 4.100 The simulation result for the Binder 4th order cumulant $U_4$ for the second kind of SW model ($J_1 = 1$, $J_3 = 1$) for $w = L$.

Note: The exact value at criticality [31] of $U_4 = 0.615$ is given by the horizontal line.
Figure 4.101 The simulation result for the order parameter $|M|$ for the second kind of SW model ($J_1 = 1, J_3 = 1$) for $w = L$.
Figure 4.102  The simulation result for the susceptibility $\chi T$ for the second kind of SW model ($J_1 = 1, J_3 = 1$) for $w = L$. 

$J_1 = J_3 = 1$, L sw spin cons 

$N=16*16$, $N=32*32$, $N=64*64$, $N=128*128$, $N=256*256$
Figure 4.103 The simulation result for the specific heat $C_H$ for the second kind of SW model ($J_1 = 1, J_3 = 1$) for $w = L$
Figure 4.104 The Binder 4th order cumulant $U_4$ scaling for the second kind of SW-model ($J_1 = 1$, $J_3 = 1$) for $w = L$.

Note: Here the scaling relation is $U_4 = U_4(tL^{1/\nu})$, where $\nu = 1$. The exact value at criticality [31] of $U_4 = 0.615$ is given by the horizontal line.
The order parameter $|M|$ scaling for the second kind of SW-model ($J_1 = 1, J_3 = 1$) for $w = L$, log-log

Note: Here the scaling relation is $|M| = L^{-\frac{3}{5}} f_M(tL^\frac{1}{\nu})$, where $\beta = \frac{1}{8}$, $\nu = 1$. The straight line has a slope of $\frac{1}{8}$. 

Figure 4.105
Figure 4.106 The susceptibility $\chi$ scaling for the second kind of SW-model ($J_1 = 1$, $J_3 = 1$) for $w = L$, log-log

Note: The scaling relation is $\chi_M^T L^{-\gamma/v} = \chi(t L^{1/v})$, where $\gamma = 1.75$, $v = 1$. The straight lines have a slope of $-1.75$. 
systems with \( w = L \) small-world spin-bonds. In Figure 4.109 and Figure 4.110, the peak positions of the susceptibility \( \chi \) and the specific heat \( C_H \) is near the Ising critical temperature, \( T_c \). But compared to the pure square lattice system, the first kind of small-world model, or the second kind of small-world model with \( w = L \) small-world spin-bonds, the values of the order parameter \( |M| \), the susceptibility \( \chi \) and the specific heat \( C_H \) are much smaller and in a large region to the critical point, these points do not fall on nearby curves. No satisfactory scaling results can be obtained for the second kind of small-world model systems with Ising scaling or mean field scaling for the system sizes studied here.

The spin-bonds can be considered as 1-dimensional spin chains. So there are \( w = \frac{L^2}{2} \) 1-dimensional spin chains in the system. These 1-dimensional spin chains have no finite temperature phase transition [53]. With \( w = \frac{L^2}{2} \) small-world spin-chains added, the number of total spins is very large. When the number of the spins on the spin-bonds is much larger than that on the regular square lattice, the 1-dimensional properties outweigh the 2-dimensional properties. That is why the order parameter \( |M| \) and the susceptibility can not be scaled by using neither Ising-like relations nor mean-field-like relations. The 1-dimensional spin-chains do not affect the Binder 4th order cumulant \( U_4 \) or the critical temperature significantly. Figure 4.111 and Figure 4.112 show the comparison of the Binder 4th order cumulant and its Ising-like scaling for the second kind of small-world model to that for the pure square lattice Ising model. The critical point only shifts a little higher from the critical temperature of the pure square lattice Ising model. The scaling data
Figure 4.107 The simulation result for the Binder 4th order cumulant $U_4$ for the second kind of SW model ($J_1 = 1$, $J_3 = 1$) for $w = L^2 / 2$

Note: The exact value at criticality [31] of $U_4 = 0.615$ is given by the horizontal line.
Figure 4.108  The simulation result for the order parameter $|M|$ for the second kind of SW model ($J_1 = 1, J_3 = 1$) for $w = \frac{L^2}{2}$.

Note:  The total number of spins grows as $L^3$, while the 'ordered' spins grows as $L^2$. 
Figure 4.109 The simulation result for the susceptibility $\chi T$ for the second kind of SW model ($J_1 = 1, J_3 = 1$) for $w = \frac{L^2}{2}$.
Figure 4.110 The simulation result for the specific heat $C_H$ for the second kind of SW model ($J_1 = 1, J_3 = 1$) for $w = \frac{L^2}{2}$.
of $U_4$ from the two models fall together. This suggests the physical small-world bonds do not affect the critical point significantly.

In summary, the second kind of small-world model exhibits Ising behavior with a small amount of long-range spin-bonds. With a large amount of long-range spin-bonds, the correspond model has a phase transition near that of the Ising critical temperature, although the Binder 4th order cumulant scales Ising-like, the order parameter and the susceptibility do not have Ising scaling relation or mean field scaling relation. This is because the number of ‘ordered’ spins scale as $L^2$ while the total number of spins scales as $L^3$. 
Figure 4.111  The comparison of the simulation result for the Binder 4th order cumulant $U_4$ for the second kind of SW model ($J_1 = 1, J_3 = 1$) for $w = L^2/2$ to that for the pure square lattice

Note: The exact values at the mean field criticality [9] of $U_4 = 0.2705201$ and at Ising criticality [31] of $U_4 = 0.615$ are given by the horizontal lines.
Figure 4.112 The scaling for the Binder 4th order cumulant $U_4$ for the second kind of SW model ($J_1 = 1$, $J_3 = 1$) for $w = \frac{L^2}{2}$ and for the pure square lattice.

Note: The exact values at mean field criticality [9] of $U_4 = 0.2705201$ and Ising criticality [31] of $U_4 = 0.615$ are given by the horizontal lines.
CHAPTER V
SUMMARY AND OUTLOOK

5.1 Summary

Three Ising models, the $z$-model, the first kind of small-world model and the second kind of small-world model, were constructed and their critical properties are studied. For all the three models, the critical point is affected by $z$, which is the average number of nearest neighbor connections of regular lattice spins. For the $z$-model, $z$ is the number of connections each spin has. For the small-world models, $z = 4 + \frac{2w}{L^2}$, where $w$ is the total small-world connections in the first kind of small-world model, or the total physical long-range bonds (small-world spin-chains) in the second kind of small-world model. The larger $z$, the higher the critical temperature. In the vicinity of the critical point, the $z$-model and the first kind of small-world model with strong long-range interaction ($\frac{J_2}{J_1} \geq 1$) and a large amount of small-world connections ($w \geq \frac{L^2}{2}$) have mean field critical properties. For the first kind of small-world model, when the long-range interaction is strong but the amount of small-world connections is small, such as $w = L$, so that the long-range connection density is vanishing, the system exhibits only a small deviation from Ising-like behavior, the mean-field properties are only due to crossovers and are very weak. When the long-range interaction is very weak, for $\frac{J_2}{J_1} = 0.01$, but the number of long-range con-
nections is large so that the long-range connection density is non-vanishing, the critical properties are Ising-like for small lattice sizes. As the long-range interaction becomes stronger, $\frac{J_2}{J_1} = 0.05$, the critical behavior deviates from Ising-like and approaches mean-field like. The crossover scaling variable is logarithmic to the system size, see Figure 4.98 and Figure 4.99. The second kind of small-world model, in which the small-world connections are physical spin-chains, behaves Ising-like with the critical temperature very close to the critical point of the pure square lattice Ising model.

The $z$-model has mean-field critical behavior. Its scaling form for $\chi T$ is given by the mean-field scaling relation predicted by Brézin and Zinn-Justin [9] rather than that postulated in [62]. Figure 4.21 and Figure 4.22 shows the Binder 4th order cumulant vs. the number of the nearest neighbor connections $z$. The critical temperature, $T_c$, which is determined by the crossing point of the Binder 4th order cumulant for large-size systems, Equation (2.159), shifts to higher temperature when $z$ becomes large. For large $z$, the relation of $T_c$ and $z$ approaches that of the Bethe lattice, Equation (3.9). For each $z$, the mean-field scaling relations of the Binder 4th order cumulant Equation (3.12), the order parameter Equation (3.15), and the susceptibility Equation (3.16), are applied to the $z$-model. These scaling relations work well for different system sizes with the same $z$, which are shown in Figure 3.25 to Figure 3.31. Also these relations work well for the same size systems with various nearest neighbor connections, $z$. Figure 3.27 shows the scaling of the Binder 4th order cumulant for various $z$ values with a system size $N = 256^2$. Figure 3.34
and Figure 3.37 show the scaling results for the order parameter and the susceptibility for $z = 3, 4, 6, 8$ with system sizes $L = 128$ and $L = 256$.

The first kind of small-world model has more complicated critical properties. For strong long-range interactions, $\frac{J_2}{J_1} \geq 1$, and the number of small-world connections proportional to the number of regular lattice spins, or say $w \propto \frac{L^2}{2}$, so that the long-range connection density is a constant, the system exhibits mean-field behaviors. From the Binder 4th order cumulant $U_4$ vs. $T$ plots, Figure 4.21 and Figure 4.21, it can be seen that the critical temperature shifts to higher temperature if $\frac{J_2}{J_1}$ or $z$ becomes larger. The mean-field scaling results of the Binder 4th order cumulant, the order parameter $|M|$, and the susceptibility $\chi$, are very good. In the scaling plots for same system size, the data from systems with different $\frac{J_2}{J_1} = 1, 4$ and/or $z = 5, 6$ fall together, see Figure 4.37, Figure 4.48 and Figure 4.52. Since the first kind of small-world model and the $z$ model both have mean field critical properties, it is valuable to compare the scaling results of same $z$ for these two models. Figure 5.1 to Figure 5.3 shows the scaling results for the Binder 4th order cumulant, the order parameter, and the susceptibility for large size systems ($L = 256$) with $z = 5$. It can be seen that for $z = 5$ the scaling curve of the order parameter fall together very well, while there are small deviations for the scaling of the Binder 4th cumulant and the susceptibility. As $z$ becomes large, or say the amount of long-range connections becomes large, the long-range interaction effects are more important compared to the short-range interaction effects, thus the systems present stronger mean field properties. In Figure 5.4 to Figure 5.6, the scaling curves of $z = 8$ fall together. Thus it can be concluded that
Ising systems with a large amount of strong long-range connections present mean field properties in the same manner.

The investigation for the case that the long-range interaction is strong and the number of small-world connections is proportional to the system size \( w \propto L \), here \( w = L \), so that the long-range connection density is vanishing, shows these systems are affected by the small-world bonds. For small size systems, the mean-field properties of a full-small world system are seen. For large size systems, the order parameter and the susceptibility scaling exhibit Ising-like behavior, Figure 4.67, Figure 4.68, Figure 4.69, and Figure 4.70, but the critical point shifts slightly to higher temperature. This indicates that even a very small amount small-world bonds can affect the behavior of the system.

In the case that the long-range interaction is very weak and the long-range connection density is non-vanishing for the first kind of small-world model, the system exhibits Ising-like properties for small system sizes, which is shown in the scaling of the Binder 4th order cumulant, Figure 4.79, the scaling of the order parameter, Figure 4.80, and the scaling of the susceptibility, Figure 4.81, where \( \frac{J_2}{J_1} = 0.01 \). As the long-range interaction becomes stronger or the system sizes become larger, the system deviates from the Ising-like behavior and exhibits mean-field properties, this can be seen from the scaling of the Binder 4th order cumulant, Figure 4.85. Although the scaling of the order parameter looks good, Figure 4.86, the scaling of the susceptibility failed by using the Ising scaling relation. The scaling prediction presented in ref. [17] for a low density or for weak SW bonds could not be seen from the simulations in this thesis. This may be caused by the finite size effects, or
Figure 5.1  The Binder 4th order cumulant $U_4$ scaling comparison of the first kind of SW-model and $z$-model for $z = 5$ for large system sizes

Note: Here $L = 256$, $J_1 = 1$, $J_2 = 1, 4$, $U_4 = f_u(tN^{1/2})$. The exact value at the mean field criticality [9] $U_4 = 0.2705201$ is given by the horizontal line.
Figure 5.2  The order parameter $|M|$ scaling comparison of the first kind of SW-model and $z$-model for $z = 5$ for large system sizes

Note: Here $L = 256$, and $J_1 = 1$, $J_2 = 1, 4$. The scaling relation is $|M| = N^{-\frac{1}{4}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 5.3 The susceptibility $\chi$ scaling comparison of the first kind of SW-model and $z$-model for $z = 5$ for large system sizes

Note: Here $L = 256$, and $J_1 = 1, J_2 = 1, 4$. The scaling relation is $\chi_{|M|} = N^{\frac{1}{2}} f_{\chi}(t N^{\frac{1}{2}})$, thus $\gamma = 1$. The straight line has a slope of $-1$. 
Figure 5.4 The Binder 4th order cumulant $U_4$ scaling comparison of the first kind of SW-model and $z$-model for $z = 8$ for large system sizes

Note: Here $L = 256$, and $J_1 = 1$, $J_2 = 1, 4$, $U_4 = f_u(t N^{1/2})$. The exact value at the mean field criticality [9] $U_4 = 0.2705201$ is given by the horizontal line.
Figure 5.5  The order parameter $|M|$ scaling comparison of the first kind of SW-model and $z$-model for $z = 8$ for large system sizes

Note: Here $L = 256$, and $J_1 = 1$, $J_2 = 1, 4$. The scaling relation is $|M| = N^{-\frac{1}{4}} f_M(tN^{\frac{1}{2}})$, thus $\beta = \frac{1}{2}$. The straight lines have slopes of $\frac{1}{2}$ and $-\frac{1}{2}$. 
Figure 5.6 The susceptibility $\chi$ scaling comparison of the first kind of SW-model and $z$-model for $z = 8$ for large system sizes

Note: Here $L = 256$, and $J_1 = 1$, $J_2 = 1, 4$. The scaling relation is $\chi_{|M|} = N^{\frac{1}{2}} f_{\chi}(tN^{\frac{1}{2}})$, thus $\gamma = 1$. The straight line has a slope of $-1$. 
from the SW bonds used here not being well approximated by the weak mean-field bonds of [17].

The crossover for the first kind of small-world model was attempted for the case that the long-range interaction is weak. The crossover scaling variable was chosen to be that proposed by Mon and Binder [46], Equation (4.11). When the effective interaction range \( R \) is calculated in the form proposed by Luijten and Binder in Ref. [39], Equation (4.16), there is no valuable scaling result found in this part of the work. This part of the work suggests that more efficient crossover relations need to be investigated.

For the second kind of small-world model, long-range physical bonds can affect the scaling properties of a system, but do not change the critical properties of the system when the system size goes to infinity. These results are consist with the studies in [53]. Systems with \( w = L \) small-world physical chains exhibit Ising-like behavior since the 1-dimensional effect is not significant. And the critical temperature approaches the value for the square lattice Ising model from the higher temperature side as the system size becomes larger, see Figure 4.100. The scaling of the Binder 4th order cumulant \( U_4 \), the order parameter \( |M| \), and the susceptibility obey the Ising scaling relations, which are shown in Figure 4.104, Figure 4.105, and Figure 4.106. But as the number of Ising spins on the small-world chains becomes larger, the systems are a mixture of 1-dimensional Ising chains and a 2-dimensional square lattice, thus clearly exhibiting neither Ising behavior nor mean-field behavior for the order parameter and the susceptibility. The 1-dimensional effect on the Binder 4th order cumulant is not significant so that the critical point is very
close to that of a pure square lattice Ising model, which can be seen from Figure 4.111. And the scaling of the Binder 4th order cumulant has Ising-like relations, shown in Figure 4.112.

In the finite-size scaling work presented in this thesis, the critical exponents used are all the exact values for Ising-like systems or mean-field systems, and scaling relations are in the expected form for Ising-like or mean-field model. There are no adjustable parameters or scaling adjustment employed, since the critical temperature estimates are obtained from crossing points of the Binder 4th order cumulant.

In summary, long-range interactions (small-world connections) affect the critical behavior of an Ising system. Systems with various long-range connections exhibit different critical behaviors, see Table 5.1. Systems with only long-range connections (the $z$-model) exhibit mean field behavior. For Ising systems that consist of both short-range (between regular nearest neighbor sites) connections and long-range (between small-world sites) connections (the first kind of small-world model), systems with a large amount of strong long-range connections exhibit mean field critical behavior while systems with a large amount of weak long-range connections exhibit Ising-like critical behavior, and systems with a small amount of long-range connections exhibits Ising-like critical behavior for small lattices. The critical point of a system with long-range interactions shifts to high temperature. When the long-range interaction becomes stronger or the number of long-range interactions becomes larger, the critical behavior exhibits a change from Ising-like to mean-field-like scaling.
| model                                | model description                                           | interaction | exponents | result figures | \( U_4 \) results | \( U_4 \) scaling | \(|M| \) scaling | \( \chi T \) scaling |
|--------------------------------------|-------------------------------------------------------------|-------------|-----------|----------------|-------------------|-------------------|------------------|-------------------|
| 2d square lattice                    | No SW cons, only short-range interactions                   | 1           | 0         | 0              | \( \frac{1}{8} \) | \( \frac{2}{4} \) | 1                | 4.53              | 2.3              | 2.4              |
| z-model                              | each spin has \( z \) nearest-neighbor cons, only long-range interaction | 0           | 1         | 0              | \( \frac{1}{2} \) | 1                | 3.3, 3.7, 3.11    | 3.25, 3.26, 3.27, 3.35, 3.36 | 3.28, 3.29, 3.32, 3.33, 3.34 | 3.30, 3.31, 3.35, 3.36, 3.37 |
| 1st kind of SW model, strong long-range interactions \((z \geq 5, w \propto L^2)\) | each spin has \( z - 4 \) SW con(s), both short- and long-range interactions | 1           | 1, 4      | 0              | \( \frac{1}{2} \) | 1                | 4.1, 4.9, 4.9, 4.10, 4.11 | 4.31, 4.32, 4.33, 4.34, 4.35 | 4.36, 4.37, 4.38, 4.39, 4.40 | 4.41, 4.42, 4.43, 4.49, 4.50, 4.51, 4.52 |
| 1st kind of SW model, strong long-range interactions \((w \leq \frac{L^2}{4}, w \propto L)\) | Each spin has 1 or 0 SW con, total \( w \) SW bonds, both short-range and long-range interactions | 1           | 1, 4      | 0              | \( \frac{1}{2} \) | \( \frac{2}{4} \) | 1                | 4.57, 4.61        | 4.65, 4.66        | 4.67, 4.68        | 4.69, 4.70       |
| 1st kind of SW model, weak long-range interactions \((z \geq 5, w \propto L^2)\) | Total \( w \) SW con(s), each spin has \( z - 4 \) SW cons, both short- and long-range interactions | 1           | 1         | 0              | \( \frac{1}{2} \) | 1                | 4.71, 4.75        | 4.79, 4.85        | 4.80, 4.86        | 4.81, no for \( J_2 = 0.05 \). |
| 2nd kind of SW model                 | Physical bonds ( spin chains as cons)                      | 1           | 0         | 1              | \( \frac{1}{8} \) | \( \frac{2}{4} \) | 1                | 4.100, 4.107      | 4.104, 4.111      | 4.105, 4.112      | 4.106            |
5.2 Outlook

Monte Carlo simulations have become a very useful tool for the study of magnetic systems. Many models have been constructed and analyzed. Finite size scaling gives a good description of the critical properties of these models. Recently, more attention has been paid to the magnetic models on complex networks and the corresponding materials. Traditionally, these systems can be modeled as random graphs (such as the $z$ model, or the model in Ref. [61]) or regular graphs (as the graphs of materials with perfect crystal structures, such as 1-dimensional ring, 2-dimensional square lattice, triangular lattice [60], and 3-dimensional cubic lattice). Recently, much more attention has been paid to networks between random graphs and regular graphs, for example, small-world networks and scale-free networks [22]. There are many discussions of models on small-world networks constructed from rewiring connections [19] [14], or from a 1-dimensional ring [56] [28] [70] [24] [25]. Although many fantastic results are obtained, there is still much that needs to be done to better understand the real magnetic material world.

Monte Carlo simulation results show that the long-range interactions affect the critical properties of the model systems. The finite-size scaling shows that the critical behavior exhibits a change from Ising-like to mean-field behavior [38] [43] [19] [10] [12] [54] [3]. The crossover properties for the first kind of small-world Ising model presented in this thesis is analyzed, but far more work is needed for better calculating the effective interaction range of the model and scaling the crossover relations.
Ising models on small-world networks in one dimension have been studied both numerically and analytically [35] [14], [20], while in two and three dimensions have been studied mostly numerically [19]. Field theory analysis [5] and renormalization group analysis [40] [36] [37] [3] have been applied to the study of Ising systems. Newman and Watts applied the renormalization group analysis to the small-world network model constructed from a 1-dimensional ring [50]. But so far no application of small-world models based on 2-dimensional lattice have been proposed.

The small-world models proposed in this thesis are different from the ones studied by others [30], [17], [51], [49], [47], [25], [70], [29], [30], [24], [19], [28], [56]. In the first kind of small-world model proposed in this thesis, all the spins in the system have the same number of long-range connections, or the difference of the connections for any two spins is no more 1. In the second kind of small-world model, the long-range bond is constructed by spin chains. Thus the critical properties may be different from the models in the above reference. Analytical discussions on these two models are expected.

The future work related to the proposed small-world models may have the following aspects. Analytical methods are expected for the study of the properties of the first kind of small-world model, such as the critical point determination and the effective interaction range calculation. These values should be obtained by classical small-world graph analysis and/or field theory. Also, a suitable crossover relation is expected for the model. For a small amount long-range connections, \( w = L \), or say vanishing density long-range connections, more work needs to be done to find the scaling relation for the Binder 4th order
cumulant as well as the order parameter and the susceptibility. For the second small-world model, it is known that the critical temperature approaches the Ising value as the systems size becomes large, but the relation of $T_c$ and $L$ is still not clear and needs to be answered analytically. Also, the 1-dimensional spin chain background needs to be expunged so that the scaling relation for the order parameter and the susceptibility can be found.

In the work in this dissertation, the long-range connection distribution effects on finite-size scaling were not considered. The long-range connection distribution affects the effective interaction range, and thus affects the finite-size scaling properties. This may lead to large fluctuations in crossover scaling for large size systems, for example, $L = 384$, which is shown in Figure 4.98 and Figure 4.99. In Luijten’s long-range interaction model [38] [43], each spin interacts with other spins within the long-range interaction region in a fixed form. Thus, the effective interaction range can be expressed as a function of the interaction radius $R_1, R_2$ and the interaction strength $J_1, J_2$. But for small-world models, the distribution of the $w$ small-world connections is not clear. For any site, there is not a fixed range within which a spin on a site interact with all other spins. The small-world connection distribution function should be in a form of $f = f(l_{sw}, N, w)$, where $l_{sw}$ is the length of small-world connections, $N$ is lattice size and $w$ is the number of small-world connections. Thus the effective range $R_{eff} = R(f, J_1, J_2)$ is a function of the small-world connection distribution function $f(l_{sw}, N, w)$ and interaction strength $J_1, J_2$. Also the small-world connection distribution may affect the values of the physical quantities being studied, such as the order parameter, the Binder 4th order cumulant and the susceptibility.
The small-world connection distribution may be obtained from small-world graph analysis and statistic simulations.

To better understand the effects of small-world connection on the critical properties, another kind of small-world Ising model may be useful. In this kind of small-world models, all small-world connections have same length, \( l_{sw} \), thus the calculation of the effective interaction range \( R_{eff} \) will be easy. Once the effects of constant length small-world connections are clear, combined with the connection distribution function, it is expected that long-range interaction effects in the first kind of small-world model proposed in this dissertation can be solved.

For the second kind of small-world model, the long-range interactions between two spins on regular sites are though small-world strands of Ising spins, or say \( 1d \) spin chains. It can be assumed that if these \( 1d \) chains are changed to equivalent long-range connection bonds with interaction strength of \( J_{2,(i,j)} \) the interaction effects will not be changed while \( 1 \)-dimensional spin chain background is expunged. In this case, the second kind of model can be modified to another kind of small-world model like the first kind of small-world model but in which the long-range interaction strength is not constant. The Hamiltonian of this kind of model can be written as

\[
\mathcal{H} = -J_1 \sum_{nn(ij)} s_i s_j - \sum_{SW(ij)} J_{2,(i,j)} s_i s_j, \tag{5.1}
\]

where \( J_{2,(i,j)} \) is long-range interaction strength between spin \( i \) and \( j \), and the second summation extends over all small-world bonds. The interaction strength \( J_{2,(i,j)} \) is a function of connection length \( l_{sw,(i,j)} \) and can be obtained from \( 1d \) Ising chain analysis or simulation.
Although in the past years, the computer calculation power increased a lot, it is still not strong enough to do simulations for every large size Ising systems. Because of computer time limitations, the work in this dissertation thesis was done for only 1 small-world network configuration for each size system for each model. In the future, with more computer power, the simulation of large size systems for the second small-world model and the modified models from the second kind of small-world model may turn to different realization. Also, the simulation and calculation for many small-world network configurations can be realized to obtain the statistic for each system size, which will provide results for a better understanding the small-world effects.
REFERENCES


